A. I. MARKUSHEVICH

THE THEORY OF ANALYTIC FUNCTIONS: A BRIEF COURSE

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About the Book

(from the Preface to the First Russian Edition)

This is a textbook in the theory of analytic functions with coverage adequate for the syllabus of physico-mathematical faculties of universities. Numerous worked-out examples of the general ideas and methods are given. In preparing the textbook, the author has drawn extensively on his book, *Theory of Functions of a Complex Variable*, translated in the USA.

About the Author

Aleksei Ivanovich MARKUSHEVICH (1908-1979) received his degree of Doctor of Physico-Mathematical Sciences from Moscow State University in 1944. He served as Professor at Moscow University and was a Full Member of the Academy of Pedagogical Sciences, USSR, from 1967. He was also First Deputy Minister of Education in the Russian Federation. Professor Markushevich's research activities covered a wide range of topics in pedagogy, history of science, and mathematics. His main interest lay in the theory of functions of a complex variable. He was the author of many textbooks in mathematics, popular-science booklets, and numerous articles on the history of function theory, the theory of entire functions, and the theory of conformal mapping. His fundamental course, *Theory of Functions of a Complex Variable*, was published in the United States by Prentice-Hall in three volumes. Professor Markushevich was Chief Editor of the *Encyclopedia of Elementary Mathematics* and the *Children's Encyclopedia*.

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Translated from the Russian
by
Eugene Yankovsky

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А. И. МАРКУШЕВИЧ

КРАТКИЙ КУРС ТЕОРИИ АНАЛИТИЧЕСКИХ ФУНКЦИЙ

ИЗДАТЕЛЬСТВО
«НАУКА»
МОСКВА

PREFACE TO THE FOURTH RUSSIAN EDITION

Since the last Russian edition (1966) several changes have taken place in the curriculum of mathematical faculties of universities. The main change is that the number of hours devoted to mathematics proper has increased. In view of the new requirements and the requests of readers studying the subject on their own, I have introduced changes into or rewritten completely several sections of the book. For one, I have included Runge's theorem on the expansion of an analytic function into a series of polynomials, expanded the sections on the maximum modulus principle, the order and type of an entire function, and the product expansion of an entire function, included the Phragmén-Lindelöf principle as applied to the study of the Phragmén-Lindelöf function for an entire function, and expounded on the general properties of elliptic functions. I have also introduced Weierstrass's $\wp$ and $\wp'$ functions, given the notion of the analytic continuation along a curve and proof of the monodromy theorem, examined the Borel transformation as applied to analytic continuation and the problem of inversion of ultraelliptic integrals, and introduced Schwarz's modular function in order to prove Picard's first theorem. Finally, in Sec. 10.7 I have included the generalized argument principle, which simplifies the proof of the theorem on correspondence between boundaries.

To keep the book down to a reasonable size, I have excluded from this edition the plane problem of hydrodynamics, which before was at the end of the book. The interested reader may turn to the special literature. The bibliography has been reviewed and brought up to date.

A.I. Markushevich

FROM THE PREFACE TO THE FIRST RUSSIAN EDITION

This is a textbook in the theory of analytic functions with coverage adequate for the syllabus of physico-mathematical faculties of universities. Numerous worked-out examples of the general ideas and methods are given. A small part of the material is not obligatory and only supplements the basic course. The interested reader may turn to the bibliography on the subject given at the end of the book.

In preparing this textbook I have drawn extensively on my other book, *The Theory of Analytic Functions* [English translation: A.I. Markushevich, *Theory of Functions of a Complex Variable*, 3 vols., Prentice-Hall, Englewood Cliffs, N.J., 1965-7].

A.I.M.

TO THE READER

Mir Publishers would be grateful for your comments on the content, translation and design of this book. We would also be pleased to receive any other suggestions you may wish to make.

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0

INTRODUCTION

0.1

THE THEORY OF ANALYTIC FUNCTIONS

The subject of this book has a double name: the theory of analytic functions and the theory of functions of a complex variable. Each name emphasizes only one aspect of the problem, since we will study analytic functions of a complex variable.

The function $f(x)$ of a real variable x defined in some interval (a, b) (finite or infinite) is called *analytic* in this interval if in the neighborhood of each point x_0 it can be represented in the form of a series in integral nonnegative powers of $x - x_0$:

$$f(x) = A_0 + A_1(x - x_0) + A_2(x - x_0)^2 + \ldots + A_n(x - x_0)^n + \ldots .$$

An arbitrary polynomial and the functions e^x, $\sin x$, and $\cos x$ are analytic functions on the entire number axis; each rational function and the functions $\tan x$, $\cot x$, $\sec x$, and $\operatorname{cosec} x$ are analytic in intervals that do not contain points at which the corresponding function is not defined (becomes infinite); and the function $\ln x$ is analytic inside the interval $(0, \infty)$. All these statements can easily be verified. For instance, if $x_0 > 0$, then

$$\ln x = \ln x_0 + \ln\left(1 + \frac{x - x_0}{x_0}\right) = \ln x_0 + \sum_{1}^{\infty} (-1)^{n-1} \frac{(x - x_0)^n}{n x_0^n}$$

at

$$|x - x_0| < |x_0|.$$

The sum, difference, product, and ratio of analytic functions (inside an interval where the denominator does not vanish) are analytic functions; also the derivative and integral of an analytic function is analytic. With some reservations the following statements are true: (a) the inverse of an analytic function is analytic; and (b) if $A_j(x)$ $(j = 0, 1, \ldots, n)$ are analytic, then the function

$f(x)$ defined by

$$A_0(x) + A_1(x) f(x) + \ldots + A_n(x) [f(x)^n] = 0$$

and also by

$$A_0(x) + A_1(x) \frac{df(x)}{dx} + \ldots + A_n(x) \frac{d^n f(x)}{dx^n} = 0$$

is analytic.

Proceeding from these propositions, it is easy to understand that the most important functions used in calculus, geometry, mechanics, and physics are analytic functions. Indeed, not only the aforementioned elementary functions are analytic, but the gamma function, cylindrical (Bessel) functions, elliptic functions and many others are also analytic in appropriate intervals. This explains why analytic functions play such a big role in mathematics and its applications. At the same time it becomes clear why we can isolate the subject of analytic functions as a separate branch of mathematics.

0.2

ANALYTIC FUNCTIONS OF A COMPLEX VARIABLE

Already in studying the simplest analytic function, the polynomial

$$f(x) = a_0 + a_1 x + \ldots + a_n x^n \quad (a_n \neq 0),$$

it proves expedient to consider it as a function of a complex variable.

Indeed, only such an approach reveals that this function takes on any value (e.g. zero) for n values of x (some of which may coincide). This leads to the basic corollary that a polynomial can be expanded into a product of linear factors:

$$f(x) = a_n (x - x_1)(x - x_2) \ldots (x - x_n),$$

and other corollaries related to this one.

Naturally, in studying polynomials as functions of a complex variable we can assume the coefficients to be arbitrary complex numbers. Similarly, when studying the most general analytic functions of the complex variable $z = x + iy$, we use the power series $\sum_0^\infty A_n (z - z_0)^n$, the coefficients $A_0, A_1, \ldots, A_n, \ldots$ and the number z_0 being complex numbers. A function $f(z)$ is said to be *analytic on a set of points in the complex plane* (representing complex numbers) if in a neighborhood of every point of this domain it can be represented by a power series in $z - z_0$:

$$f(z) = \sum_0^\infty A_n (z - z_0)^n.$$

It appears then that we can extend to functions of a complex variable the main notions of calculus, such as the notion of a derivative $f'(z)$ and an integral $\int_L f(z)\,dz$ evaluated along a planar curve L.

In our course we will prove the following fundamental facts. *For a function $f(z)$ to be analytic inside a circle in the complex z-plane it is necessary and sufficient that the following four conditions hold true:*

(a) *the function $f(z)$ has a derivative $f'(z)$ at each point inside the circle;*

(b) *if $f(z) = u(x, y) + iv(x, y)$, where $u(x, y)$ and $v(x, y)$ are real-valued functions of real variables x and y, then $u(x, y)$ and $v(x, y)$ are twice differentiable functions that satisfy the Laplace diffe ential equation*

$$\frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2\varphi}{\partial y^2}=0$$

and are coupled through the following equations:

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x};$$

(c) *the function $f(z)$ is continuous inside the circle, and the integral of this function evaluated along every closed curve inside the circle is zero;*

(d) *inside any concentric circle of a smaller radius the function $f(z)$ can be approximated by polynomials with an arbitrarily high accuracy.*

These propositions form the basis of the theory of analytic functions of a complex variable. Each of the properties (a)-(d) can be used to define an analytic function of a complex variable. In this course we will use the definition based on property (a); only subsequently will we show that this definition is equivalent to that in terms of a power series.

Note that many applications of analytic function theory in physics and mechanics are based on proposition (b). For instance, the so-called planar problems of thermal and electrical equilibrium and the problem of laminar flow of fluids around flat profiles lead to the Laplace equation, from the various solutions of which all analytic functions can be built.

1

COMPLEX NUMBERS AND THEIR GEOMETRIC INTERPRETATION

1.1

THE INTERPRETATION OF COMPLEX NUMBERS ON A PLANE

The theory of complex numbers is usually presented in higher algebra courses.* Here we will recall only the main definitions and results of this theory and augment them somewhat in our future exposition.

Each complex number c has the form $a + ib$, where a and b are real numbers and i is the so-called *imaginary unit*; $a = \operatorname{Re} c$ is called the *real part* of c, and $b = \operatorname{Im} c$ the *imaginary part* (Re comes from the Latin word *realis* and Im from *imaginarius*). Two complex numbers c' and c'' are equal if and only if $\operatorname{Re} c' = \operatorname{Re} c''$ and $\operatorname{Im} c' = \operatorname{Im} c''$. If $\operatorname{Im} c = 0$, the number $c = \operatorname{Re} c$ is real; if $\operatorname{Im} c \neq 0$, the number c is called an *imaginary number*, and if, in addition, $\operatorname{Re} c = 0$ it is called *pure* (or *purely*) *imaginary*.

To interpret complex numbers geometrically, we assign to a plane rectangular coordinate axes and consider each point $M(x, y)$ in the plane as the image of a complex number $z = x + iy$ called the *affix* of point M. This condition uniquely relates the set of all points in a plane and the set of all complex numbers. The set of all real numbers is then represented by the abscissa axis, which for this reason is called the *real axis*; the set of all imaginary numbers is represented by the set of points not lying on the real axis; and the set of all pure imaginary numbers by the ordinate axis, called the *imaginary axis* (there is one point on the imaginary axis that represents a real number, point zero). The plane as a whole is called the *complex* (*number*) *plane* (*Gauss-Argand plane* or *Gaussian plane*), as well as the z-plane, the w-plane, etc. depending on what letter denotes the complex numbers.

The terms "complex number $x + iy$" and "point $x + iy$" are used here as synonyms.

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* See, for example, A. G. Kurosh, *Higher Algebra*, Mir Publishers, Moscow, 1972 (reprinted in 1975 and 1980).

For the geometric interpretation of $z = x + iy$, aside from point (x, y) we can use a vector whose projections on the coordinate axes are x and y; the beginning can be placed at an arbitrary point (Fig. 1). For this reason we use the terms "complex number" and "vector" interchangeably. The length $|z|$ of a vector is called the *modulus* of complex number z, and the angle Arg z between the positive real axis and the vector (we assume here that $z \neq 0$) is called the *argument* (*amplitude*) of the complex number. (The argument is known to within a term that is an integral multiple of 2π.) There is one and only one value α of the argument that satisfies the condition $-\pi < \alpha \leqslant \pi$; it is called the *principal value of the argument* and is denoted arg z. Obviously

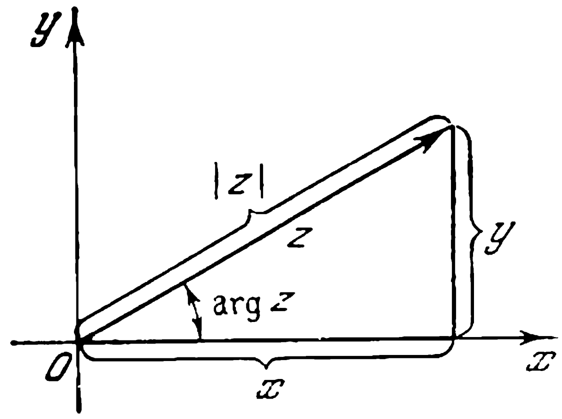

Fig. 1

$$\operatorname{Arg} z = \arg z + 2k\pi,$$

where k is any integer.

We note the following additional relationships:

if $z = x + iy$, then $|z| = \sqrt{x^2 + y^2}$;

$$\arg z = \arctan \frac{y}{x} \quad \text{at } x > 0,$$

$$\arg z = \arctan \frac{y}{x} + \pi \text{ at } x < 0 \text{ and } y \geqslant 0,$$

$$\arg z = \arctan \frac{y}{x} - \pi \text{ at } x < 0 \text{ and } y < 0.$$

The real and imaginary parts of z are determined in terms of the modulus and argument thus: $\operatorname{Re} z = |z| \cos \operatorname{Arg} z$ and $\operatorname{Im} z = |z| \sin \operatorname{Arg} z$. We can therefore represent a complex number z in the form

$$z = |z| (\cos \operatorname{Arg} z + i \sin \operatorname{Arg} z),$$

which is known as the *trigonometric form* of z.

The complex numbers $x + iy$ and $x - iy$ are called *conjugate complex*. If one is denoted by z, the other is denoted by $\bar{z}$. Obviously the points z and $\bar{z}$ are symmetric with respect to the real axis. For this reason $|z| = |\bar{z}|$; also $\arg z = -\arg \bar{z}$ if z is not a negative real number (otherwise $\arg z = \arg \bar{z} = \pi$).

1.2
OPERATIONS ON COMPLEX NUMBERS

Addition and multiplication of complex numbers are determined via the following relations:

$$(a_1 + ib_1) + (a_2 + ib_2) = (a_1 + a_2) + i(b_1 + b_2),$$
$$(a_1 + ib_1)(a_2 + ib_2) = (a_1a_2 - b_1b_2) + i(a_1b_2 + a_2b_1),$$

and subtraction and division as the corresponding inverse operations. These definitions lead to the following corollaries: addition and multiplication obey the commutative and associative laws, multiplication obeys the distributive law with respect to addition, the

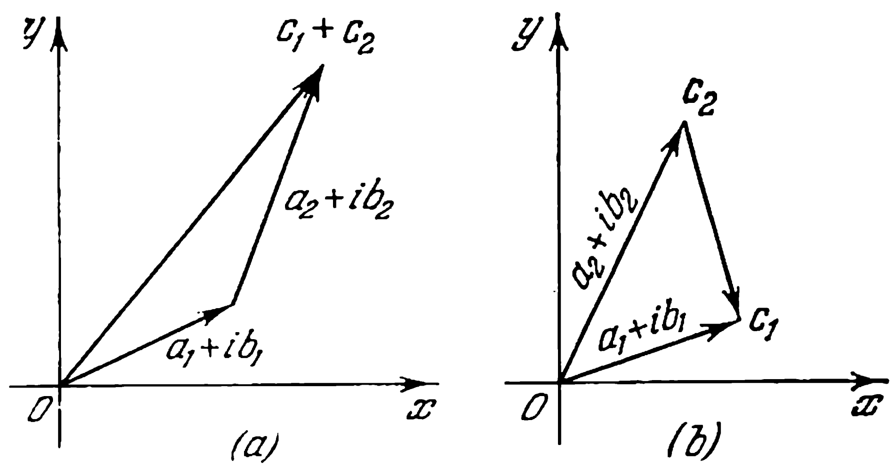

Fig. 2

product of two complex numbers is zero if and only if at least one of the multiplicands is zero, subtraction is always possible, and division is possible only if the denominator is nonzero. The properties imply that complex numbers form a field. One particular case of multiplication: if $c = a + ib$, then $\bar{c} \times c = a^2 + b^2 = |c|^2$, whence $|c| = \sqrt{c \times \bar{c}}$.

Geometrically, addition of $c_1 = a_1 + ib_1$ and $c_2 = a_2 + ib_2$ is done just as for vectors (Fig. 2*a*). The difference $c_1 - c_2$ is represented by a vector with the beginning at point c_2 and the end at point c_1 (Fig. 2*b*). This implies that the distance between two points, $c_1 = a_1 + ib_1$ and $c_2 = a_2 + ib_2$, is equal to the modulus of the difference of c_1 and c_2: $\rho(c_2, c_1) = |c_1 - c_2|$. Moreover, we note the following important inequalities for the modulus of a sum and difference:

$$|c_1 + c_2| \leqslant |c_1| + |c_2|, \quad |c_1 - c_2| \geqslant ||c_1| - |c_2||;$$

the equality signs are present only if c_1 and c_2 are collinear vectors pointing in the same direction.

The first inequality can be extended to any number of addends:

$$|c_1 + c_2 + \ldots + c_n| \leqslant |c_1| + \ldots + |c_n|;$$

here too equality is present only if the n vectors $c_1, c_2, \ldots, c_n$ are collinear and point in the same direction.

To interpret multiplication geometrically, we write c_1 and c_2 in trigonometric form:

$$c_1 = |c_1| (\cos \operatorname{Arg} c_1 + i \sin \operatorname{Arg} c_1),$$
$$c_2 = |c_2| (\cos \operatorname{Arg} c_2 + i \sin \operatorname{Arg} c_2);$$

then by definition of multiplication we have

$$c = c_1 c_2 = |c_1| |c_2| [\cos (\operatorname{Arg} c_1 + \operatorname{Arg} c_2) + i \sin (\operatorname{Arg} c_1 + \operatorname{Arg} c_2)].$$

Whence

$$|c_1 c_2| = |c_1| |c_2| \quad \text{and} \quad \operatorname{Arg} (c_1 c_2) = \operatorname{Arg} c_1 + \operatorname{Arg} c_2$$

(the last relationship is to be understood in the sense that by adding the possible values of $\operatorname{Arg} c_1$ and $\operatorname{Arg} c_2$ we arrive at a set of numbers that coincides with the set of values of $\operatorname{Arg} (c_1c_2)$). Geometrically, multiplication of c_1 by c_2 ($c_1 \neq 0$ and $c_2 \neq 0$) means that vector c_1 is stretched by $|c_2|$ times and rotated about its origin by an angle $\operatorname{Arg} c_2$. For the ratio $c_1 \div c_2 = c_1/c_2$ ($c_1 \neq 0$ and $c_2 \neq 0$) we find that

$$|c_1 \div c_2| = |c_1| \div |c_2| \quad \text{and} \quad \operatorname{Arg} (c_1 \div c_2) = \operatorname{Arg} c_1 - \operatorname{Arg} c_2.$$

The last relationship implies that the angle between vectors c_1 and c_2 from c_2 to c_1 counterclockwise (and defined to within a term that is an integral multiple of 2π) is equal to $\operatorname{Arg} (c_1/c_2)$:

$$\widehat{c_2, c_1} = \operatorname{Arg} \frac{c_1}{c_2}.$$

From the multiplication rule it follows that

$$c^n = |c|^n (\cos n \operatorname{Arg} c + i \sin n \operatorname{Arg} c),$$

where n is a positive integer; obviously this formula is valid when $n = 0$ ($c^0 = 1$). Noting that $c^{-n} = 1/c^n$, we obtain

$$c^{-n} = |c|^{-n} [\cos (-n \operatorname{Arg} c) + i \sin (-n \operatorname{Arg} c)].$$

Hence, the following formula is true for any m:

$$c^m = |c|^m (\cos m \operatorname{Arg} c + i \sin m \operatorname{Arg} c).$$

If p and q are integers, $q \geqslant 2$, and p/q is irreducible, the right-hand side of the formula

$$c^{\frac{p}{q}} = \sqrt[q]{c^p} = \sqrt[q]{|c|^p}_{+} \left[\cos \left(\frac{p}{q} \operatorname{Arg} c\right) + i \sin \left(\frac{p}{q} \operatorname{Arg} c\right)\right],$$

where $\sqrt[q]{|c|^p}_{+}$ denotes the positive value of $|c|^{p/q}$, gives the q different values of $c^{p/q}$.

To obtain all these values, it is sufficient to fix one value of Arg c, say φ, and substitute for Arg c into the right-hand side the following n values: φ, $\varphi + 2\pi$, . . ., $\varphi + (q - 1)\, 2\pi$.

1.3
THE LIMIT OF A SEQUENCE

A sequence of complex numbers $\{c_n = a_n + ib_n\}$ is said to *converge to a limit* $c = a + ib$ (in shorter form: $\lim\limits_{n\to\infty} c_n = c$, or $c_n \to c$ as $n \to \infty$) if for every positive number ε it is possible to indicate a positive number $N(\varepsilon)$ such that $|c_n - c| < \varepsilon$ for $n > N(\varepsilon)$. Since $|a_n - a| \leqslant |c_n - c| < \varepsilon$ and $|b_n - b| \leqslant |c_n - c| < \varepsilon$ for $n > N(\varepsilon)$, we find that $\lim\limits_{n\to\infty} a_n = a$ and $\lim\limits_{n\to\infty} b_n = b$. Hence, the last two equalities are consequences of the fact that $\lim\limits_{n\to\infty} (a_n + ib_n) = a + ib$. Conversely, if $\lim\limits_{n\to\infty} a_n = a$ and $\lim\limits_{n\to\infty} b_n = b$, then $|a_n - a| < \varepsilon/\sqrt{2}$ and $|b_n - b| < \varepsilon/\sqrt{2}$ for $n > N_1(\varepsilon)$; whence

$$|a_n + ib_n - (a + ib)| = |c_n - c| = \sqrt[+]{(a_n - a)^2 + (b_n - b)^2} < \varepsilon$$

for $n > N_1(\varepsilon)$, i.e. $\lim\limits_{n\to\infty} c_n = c$.

Therefore, $\lim\limits_{n\to\infty} (a_n + ib_n) = a + ib$ *is equivalent to two relationships:* $\lim\limits_{n\to\infty} a_n = a$ *and* $\lim\limits_{n\to\infty} b_n = b$. This remark enables us to apply the entire theory of sequences of real numbers onto sequences of complex numbers. For instance, the following necessary and sufficient condition for convergence (*Cauchy's test*) is valid: *for every positive* ε *there is an* $N(\varepsilon)$ *such that* $|c_{n+p} - c_n| < \varepsilon$ *for* $n > N(\varepsilon)$ *and* p *an arbitrary positive integer*. Moreover, if $\lim\limits_{n\to\infty} c'_n = c'$ and $\lim\limits_{n\to\infty} c''_n = c''$, then

$$\lim_{n\to\infty} (c'_n \pm c''_n) = c' \pm c'', \quad \lim_{n\to\infty} (c'_n \cdot c''_n) = c'c'', \quad \lim_{n\to\infty} \frac{c'_n}{c''_n} = \frac{c'}{c''}$$

(the last formula holds if $c''_n \neq 0$, $n = 1, 2, \ldots$, and $c'' \neq 0$).

Let us call the set of points lying inside a circle of radius ρ centered in the point c the *ρ-neighborhood of point c*. Obviously point z belongs to this neighborhood if and only if $|z - c| < \rho$. We can now give the limit of a sequence $\{c_n\}$ the following geometric meaning: *a sequence $\{c_n\}$ is said to converge to its limit c if for every positive ε all points of the sequence starting from a certain number belong to the ε-neighborhood of the point c.*

We suggest that reader prove that $\lim\limits_{n\to\infty} c_n = c$ always implies that $\lim\limits_{n\to\infty} |c_n| = |c|$. If, in addition, $c \neq 0$, there exists a sequence of arguments of c_n whose limit is one of the values of the argument of c. As such a sequence we can take the sequence of the principal values of the arguments except when $c < 0$ and when there is an infinite number of points from $\{c_n\}$ lying above and below the real axis. The above-mentioned property of the arguments of the sequence $\{c_n\}$ can symbolically be written as $\lim\limits_{n\to\infty} \operatorname{Arg} c_n = \operatorname{Arg} c$. Conversely, if the last condition is met and in addition $\lim\limits_{n\to\infty} |c_n| = |c|$, then $\lim\limits_{n\to\infty} c_n = c$.

1.4

INFINITY AND THE STEREOGRAPHIC PROJECTION

In studying analytic functions we will need, in addition to *proper* (finite) complex numbers, one more *improper* (infinite) complex number denoted by ∞; this number is called *infinity*, or the *point at infinity*. The following definitions and rules will show us how to deal with the point at infinity.

We call the exterior of a circle of radius ρ centered at the origin the ρ-*neighborhood of the point at infinity* ∞. Obviously, point z belongs to this neighborhood if and only if $|z| > \rho$. We say that a sequence $\{c_n\}$ *converges* to ∞ (in short form: $\lim\limits_{n\to\infty} c_n = \infty$) if for every positive ρ all points starting from a certain number belong to the neighborhood $|z| > \rho$ of the point at infinity. In other words, for every $\rho > 0$ there exists a positive $N(\rho)$ such that $|c_n| > \rho$ for $n > N(\rho)$; hence $\lim\limits_{n\to\infty} c_n = \infty$ is equivalent to $\lim\limits_{n\to\infty} |c_n| = +\infty$. We also note that when $c_n \neq 0$, the condition $\lim\limits_{n\to\infty} |c_n| = \infty$ is equivalent to $\lim\limits_{n\to\infty} (1/c_n) = 0$.

For the improper complex number we do not introduce the notions of the real and imaginary parts or the notion of the argument; strictly speaking, these notions are meaningless in this case (note that the notion of the argument is meaningless for 0 as well). As to the modulus of complex number ∞, we use the symbol $+\infty$, i.e. $|\infty| = +\infty$.

The definition of the point at infinity is used to establish the meaning of the following expressions with ∞ and proper complex numbers a and α $(\alpha \neq 0)$:

$$\infty \pm a = a \pm \infty = \infty, \quad \infty \cdot \alpha = \alpha \cdot \infty = \infty \cdot \infty = \infty,$$

$$\frac{a}{\infty} = 0, \quad \frac{\infty}{a} = \infty, \quad \frac{\alpha}{0} = \infty.$$

The operations $\infty \pm \infty$, $0 \cdot \infty$, $0/0$, and ∞/∞ are declared without meaning.

To interpret the number ∞ geometrically, we use the representation of complex numbers by the points of a sphere.

To this end from point O of the complex z-plane as center we describe a sphere of unit radius (Fig. 3). For the sake of visualization we introduce geographic terminology. The circle along which the sphere intersects the complex plane is called the *equator*, the straight line that passes through O perpendicular to the z-plane the *axis of the sphere*, and points N and S at which the axis intersects the sphere the *North* and *South Poles*, respectively. Other terms are: the *Northern*

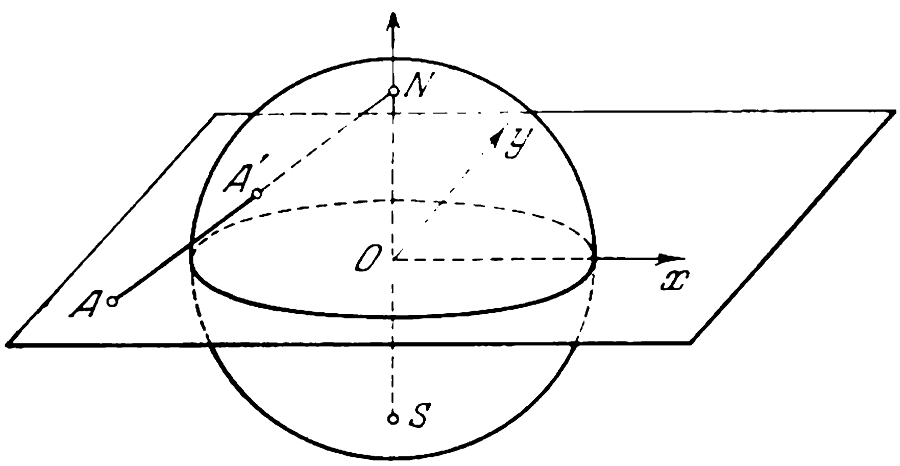

Fig. 3

and *Southern Hemispheres*, *meridians*, *parallels*, *latitude* φ, and *longitude* λ. Latitude is reckoned from the equator from $-\pi/2$ (South Pole) to $\pi/2$ (North Pole). Longitude is reckoned from the equatorial plane from the positive semiaxis Ox from $-\pi$ (excluding $-\pi$) to π (including π); the positive direction here is considered the counterclockwise direction if one looks at the equator from the North Pole. Let us now connect point N with various points on the sphere by straight lines emerging from N and mark the points where these lines meet the z-plane. This projection (a central projection with center at N) is called *stereographic*; this method of projection has been used for centuries first in astronomy and then in geography to represent the celestial and terrestrial spheres on a plane. By means of stereographic projection each point on the sphere (except N) may be considered as the image of an appropriate point in the plane and hence as the image of the complex number corresponding to this point in the plane. Let us see how the latitude and longitude of a point on the sphere that represents a complex number $z = r(\cos\alpha + i\sin\alpha)$ $(\alpha = \arg z)$ is connected with the modulus and argument of this number.

From Fig. 4 we can see that $\widehat{ONA'} = \pi/4 + \varphi/2$ and hence $r = \tan(\pi/4 + \varphi/2)$; moreover, it is obvious that $\alpha = \lambda$. We see that

$\varphi = 2 \arctan r - \pi/2$ and $\lambda = \alpha$. If for the sequence $\{z_n\}$ in which $|z_n| = r_n$ the condition $\lim_{n\to\infty} z_n = \infty$ is met, then $\lim_{n\to\infty} r_n = +\infty$ and hence $\lim_{n\to\infty} \varphi_n = \lim_{n\to\infty} (2 \arctan r_n - \pi/2) = \pi/2$. Therefore, the points on the sphere representing the numbers z_n converge toward the North Pole N. The converse is also true: if $\varphi_n \to \pi/2$ (whatever the values of the longitudes λ_n), then $r_n = \tan(\pi/4 + \varphi_n/2) \to +\infty$ and hence $\lim_{n\to\infty} z_n = \infty$. It is natural then to consider point N as the image of the point at infinity. This is in good agreement with the fact that the neighborhood $|z| > \rho$ of the point at infinity in the plane is depicted on the sphere by the near-polar region $\varphi > 2 \arctan \rho - \pi/2$; as $\rho \to \infty$ this region converges toward the North Pole.

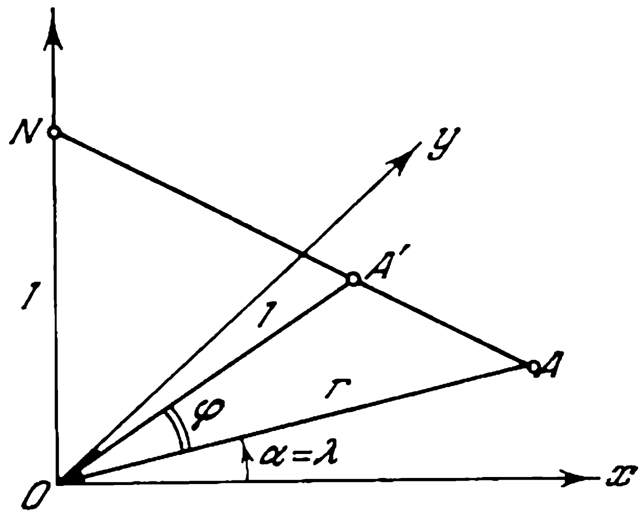

Fig. 4

The complex plane to which we have mentally added the unique point at infinity is called the *extended complex plane*, or simply the *extended plane*. Geometrically the pictorial representation of an extended plane is the entire surface of the sphere. The complex plane formed only from proper (finite) points is called the *finite complex plane*, or simply the *finite plane*. It is readily seen that the finite plane is represented by a sphere without one point, namely point N.

Let us reflect the sphere by placing a mirror in its equatorial plane. The sphere transforms into itself, i.e. the Northern Hemisphere transforms into the Southern (and vice versa), the North Pole becomes the South Pole (and vice versa), and the equator transforms into itself. In general, each point with the geographic coordinates (φ, λ) becomes a point $(-\varphi, \lambda)$.

This mapping of the sphere onto itself corresponds to the mapping of the extended plane onto itself, a mapping in which point z with coordinates $r = \tan(\pi/4 + \varphi/2)$ and $\alpha = \lambda$ transforms into a point z' with coordinates $r' = \tan(\pi/4 - \varphi/2)$ and $\alpha' = \lambda = \alpha$. Obviously, z and z' are related thus: $\bar{z} \times z' = 1$, i.e. $z' = 1/\bar{z}$. This mapping transforms the exterior of a unit circle into its interior (and vice versa); in particular, point ∞ is transformed into point 0 (and vice versa). The unit circle is mapped onto itself. The transformation $z' = 1/\bar{z}$ may be considered as the symmetry transformation of the extended plane with respect to a unit circle or the mirror reflection in the unit circle. This approach is justified if we consider what happens on the sphere in the course of the transformation. In

Sec. 3.8 we will give a more general definition of the symmetry transformation of the extended plane with respect to an arbitrary circle in the plane.

1.5

SETS OF POINTS IN A PLANE

We will now recall some definitions and properties of sets of points in a plane known from the calculus course and augment them for further exposition.

A point z_0 is called a *limit point* for a set E if every neighborhood of this point contains an infinite set of points from E.

A set E of points in a plane is said to be *bounded* if all its points are contained inside a circle with the center at the origin of coordinates. A set F (bounded or unbounded) is said to be *closed* if there is no point not belonging to F that is a limit point for F. In other words, a closed set either has no limit points or contains all its limit points.

Any infinite bounded set E has at least one limit point (the **Bolzano-Weierstrass theorem**). If an infinite set E is unbounded, there are only two possibilities: either a circle $|z| \leqslant R$ will contain an infinite set of points from E and, hence, the limit point of this set, or each such circle will have only a finite number of points from E and, hence, in any neighborhood $|z| > R$ of the point at infinity there will be an infinite set of points (which implies that ∞ is the limit point of E). Therefore, *in the extended plane each infinite set has at least one limit point* (finite or at infinity).

Let $\{K\}$ be a set of circles covering a bounded closed set F, i.e. such circles that for each $z \in F$ there exists at least one circle from $\{K\}$ containing this point. Then from the set $\{K\}$ we can select a finite number of circles $K_1, K_2, \ldots, K_n$ covering F (the **Heine-Borel theorem**).

Let E be a set of points in a plane. The lower bound of distances from a point ζ (not in E) to all possible points in E is called the *distance between point ζ and set E*, i.e. $\rho(\zeta, E) = \inf |\zeta - z|$, $z \in E$. If $\rho(\zeta, E) = 0$, then either $\zeta \in E$ or $\zeta \notin E$, but in the latter case E contains points lying arbitrarily close to ζ, i.e. ζ is a limit point of E. When E is closed, a point not belonging to E cannot be its limit point. *For this reason from $\zeta \notin E$ it follows that* $\rho(\zeta, E) > 0$. Let E and F be two sets of points. The lower bound of distances between all possible pairs of points z' and z'' such that $z' \in E$ and $z'' \in F$ is called the *distance between the sets*, i.e. $\rho(E, F) = \inf |z' - z''|$. The distance between E and F may be zero even if E and F have no common points. But *if both sets, or at least one, are closed, then from the fact that E and F have no common points it follows that* $\rho(E, F) > 0$.

Indeed, if $z' \in E$, then $z' \notin F$ and hence $\rho(z', F) > 0$. Take a circle of radius $\rho(z', F)$ centered at z'; inside this circle there will be no

points from F. The set of circles centered at the same points and having radiuses that are half of $\rho(z', F)$ cover set E. According to Heine-Borel's theorem, there is a finite number of circles $K_1, K_2, \ldots, K_n$ with centers at $z'_1, z'_2, \ldots, z'_n$ and radiuses $\rho(z'_1, F)/2, \rho(z'_2, F)/2, \ldots, \rho(z'_n, F)/2$ covering E. We denote the smallest of these radiuses by δ ($\delta > 0$). Let $z' \in E$. Then $z' \in K_j$, and since a concentric circle of twice the radius, $2\rho(z'_j, F)/2$, does not contain a single point from F, for every point $z'' \in F$ we have

$$|z' - z''| \geqslant \frac{1}{2}\rho(z'_j, F) \geqslant \delta.$$

Whence

$$\rho(z', z'') = \inf |z' - z''| \geqslant \delta > 0,$$

which is what we set out to prove.

A point of a set E is called an *interior point* (with respect to this set) if there exists a neighborhood of this point contained in E. A set E that consists only of interior points is called an *open set*. Points that are limit points of an open set E and do not belong to it are called *boundary points*, and the set of these points make up the *boundary* Γ of set E. A boundary is always a closed set. Another closed set is $\bar{E} = E \cup \Gamma$, which is obtained by joining the open set E and its boundary Γ and is called the *closure* of E. Still another example of a closed set is the set of all points in the plane not belonging to E. The last set splits into two subsets: the boundary Γ of E and the set E_1 of points not belonging to E and not limit points of E; the latter points are called *exterior points*. For each exterior point there exists a neighborhood that does not belong to E, and such a neighborhood contains only exterior points. Hence the set E_1 is open. An important particular case of an open set is a domain. An open set E is called a *domain* if any two points from E can be linked by a broken line that lies entirely inside E (in a particular case the broken line may be a rectangular segment). Domains are usually denoted by the letters G (for the German *Gebiet*), D (for the French *domaine*), and B (for the German *Bereich*).

Examples. (a) All the points z satisfying the inequality $|z - z_0| < \rho$ (where ρ is a fixed positive number) form a domain G, the interior of a circle with center at z_0 and radius ρ. The boundary of this domain is the circumference Γ of the circle: $|z - z_0| = \rho$. The exterior points are characterized by the inequality $|z - z_0| > \rho$. The collection of such points also form a domain G_1, the exterior of the circle. Point ∞ is likewise an exterior point in relation to G, which means that it belongs to G_1. The boundary of G_1 is Γ. In addition, every point $z \in G$ is an exterior point in relation to G_1, so that the collection of all the points exterior to G_1 coincides with G.

(b) Let Γ be $Ax + By + C = 0$ (A, B, and C are real numbers, and $A^2 + B^2 \neq 0$), which is a straight line in the z-plane. All

points that satisfy the inequality $Ax + By + C > 0$ form one domain G_1 and the points that satisfy the inequality $Ax + By + C < 0$ the other domain G_2; the two domains have a common boundary Γ and are *half-planes* (bounded by Γ). Each of the domains consists of points exterior with respect to the other. Point $z = \infty$ is a boundary point for G_1 and G_2 (Γ passes through this point).

(c) The set of points $R_1 < |z - z_0| < R_2$ is a domain (an annulus) whose boundary consists of two concentric circles Γ_1: $|z - z_0| = R_1$ and Γ_2: $|z - z_0| = R_2$. The collection of exterior points splits into two domains: the interior of the circle $|z - z_0| < R_1$ and the exterior of the circle $|z - z_0| > R_2$.

2

FUNCTIONS OF A COMPLEX VARIABLE. THE DERIVATIVE AND ITS GEOMETRIC AND HYDROMECHANICAL INTERPRETATION

2.1

A FUNCTION OF A COMPLEX VARIABLE

The idea of a function of a complex variable is a particular case of the general mathematical notion of a function. Namely, if E is a set of points in the complex z-plane and if to every value of z we ascribe one or several complex numbers w, then we say that a *function of a complex variable* z is defined on E and the values of this functions are the various values of w; in shorter form $w = f(z)$. If to each value of z there corresponds only one value of w, we say that the function is *single-valued*; but if some values of z have corresponding to them more than one value of w, the function is said to be *many-valued* (or *multivalued*). For instance, $w = z^n$ (n a positive integer), $w = |z|$, $w = \bar{z}$, $w = \operatorname{Re} z$, and $w = \operatorname{Im} z$ are single-valued functions defined in the entire (finite) plane, $w = \sqrt[n]{z}$ is a many-valued (n-valued) function defined in the entire plane, and $w = \operatorname{Arg} z$ is a many-valued (infinite-valued) function defined on the set of points different from zero. If E is on the real axis, then $z = x$ is a real variable. If, in addition, all values of w are real, we arrive at the notion of a real-valued function of a real variable, which here is a very special case of a function of a complex variable.

Let us assume that in general $z = x + iy$ and $w = u + iv$. Then the statement "the function $w = f(z)$ (e.g. single-valued) is defined on E" is equivalent to the following: "to each point from E with coordinates x and y there corresponds a real number u and a real number v". In other words, on E there are defined two real-valued functions $u = \varphi(x, y)$ and $v = \psi(x, y)$ of two real variables x and y. Hence, one complex relationship $w = f(z)$ is equivalent to two relationships, i.e. $u = \varphi(x, y)$ and $v = \psi(x, y)$. For instance, $w = z^2 = (x + iy)^2 = x^2 - y^2 + 2ixy$ is equivalent to $u = x^2 - y^2$ and $v = 2xy$.

2.2

THE LIMIT OF A FUNCTION AT A POINT

Let $w = f(z)$ be a single-valued function defined on E, and let z_0 be a limit point of E. If for a fixed complex number A and a positive ε there exists a positive $\delta(\varepsilon)$ such that $|f(z) - A| < \varepsilon$ for $|z - z_0| < \delta(\varepsilon)$, $z \in E$ (and $z \neq z_0$), then we say that $f(z)$ *tends to a limit A as z tends to z_0* and write

$$\lim_{z \to z_0,\ z \in E} f(z) = A.$$

In what follows we will always leave out $z \in E$ to simplify notation, when no ambiguity is possible.

Setting $A = B + iC$, $f(z) = u(x, y) + iv(x, y)$, and $z_0 = x_0 + iy_0$ and reasoning in the same way as we did in Sec. 1.3, we find that the above complex relationship is equivalent to two real-valued relationships:

$$\lim_{x \to x_0,\ y \to y_0} u(x, y) = B, \qquad \lim_{x \to x_0,\ y \to y_0} v(x, y) = C.$$

This remark shows that the simplest statements concerning the limits of functions of real variables transform without alteration to statements on limits of functions of a complex variable. For instance, if the functions $g(z)$ and $h(z)$ are defined on one and the same set E and if

$$\lim_{z \to z_0} g(z) = A_1, \quad \lim_{z \to z_0} h(z) = A_2,$$

then

$$\lim_{z \to z_0} [g(z) \pm h(z)] = A_1 \pm A_2,$$

$$\lim_{z \to z_0} g(z)\, h(z) = A_1 \times A_2,$$

$$\lim_{z \to z_0} \frac{g(z)}{h(z)} = \frac{A_2}{A_1}$$

(the last relationship is valid provided $A_2 \neq 0$).

In a similar manner we can consider the case where instead of a finite point we take the point at infinity. Indeed, if point $z = \infty$ is a limit point for E and if for a fixed complex number A and any positive ε there exists an $N(\varepsilon)$ such that $|f(z) - A| < \varepsilon$ for $|z| > N(\varepsilon)$ $(z \in E)$, then we say that $f(z)$ *tends to a limit A as z tends to ∞* and write

$$\lim_{z \to \infty} f(z) = A.$$

Obviously, the difference between this case and the previous one lies solely in the fact that instead of the neighborhood $|z - z_0| <$

$< \delta(\varepsilon)$ of the finite point z_0 we take the neighborhood $|z| > N(\varepsilon)$ of the point at infinity.

Finally, if z_0 is any limit point of set E (finite or at infinity) and if for every positive N we can indicate a neighborhood of this point such that the inequality $|f(z)| > N$ holds for any z belonging to this neighborhood ($z \in E$ and $z \neq z_0$), then we say that $f(z)$ *tends to* ∞ *as* z *tends to* z_0 and write

$$\lim_{z \to z_0} f(z) = \infty.$$

The three particular cases of a limit of a function can be summarized by the following general definition. *Let z_0 be a limit point of set E (finite or at infinity) and A a complex number (proper or improper). If for each neighborhood U of point A there exists a neighborhood U_0 of point z_0 such that $f(z)$ belongs to U if z belongs to U_0 (and also $z \in E$ and $z \neq z_0$), then it is said that $f(z)$ tends to the limit A as z tends to z_0*:

$$\lim_{z \to z_0,\ z \in E} f(z) = A.$$

With such a general definition of a limit, when it is possible that $A = \infty$, we cannot use the theorems on the sum, difference, product and ratio of functions without reservations since $\infty \pm \infty$, $0 \cdot \infty$, and ∞/∞ have no meaning.

2.3
CONTINUITY

If a limit point z_0 of the set E (finite or at infinity) belongs to this set and if for a function $f(z) = u(x, y) + iv(x, y)$ defined on E the condition

$$\lim_{z \to z_0,\ z \in E} f(z) = f(z_0) \quad (f(z_0) \neq \infty)$$

is satisfied, then $f(z)$ is said to be *continuous at point* z_0 (on set E).

If $f(z)$ is continuous at each point of E, then $f(z)$ is said to be *continuous on* E.

Recalling the results of Sec. 2.2, we can see that continuity of $f(z)$ at $z_0 = x_0 + iy_0$ is equivalent to two relationships:

$$\lim_{x \to x_0,\ y \to y_0} u(x, y) = u(x_0, y_0), \qquad \lim_{x \to x_0\ y \to y_0} v(x, y) = v(x_0, y_0),$$

which express the continuity of two real-valued functions $u(x, y)$ and $v(x, y)$ at the same point.

Hence, *a function of a complex variable is continuous at point z_0 if and only if its real and imaginary parts considered as functions of real variables x and y are continuous at the same point.*

This implies that many properties of continuous functions of two real variables can be carried over to continuous functions of a complex variable.

For instance, the sum, difference, product, and ratio of two continuous functions are continuous functions (in the case of a ratio we exclude the points at which the denominator vanishes). Moreover, if a function $w = f(z)$ is continuous on a set E and its values belong to a set F on which a function $\zeta = \varphi(w)$ is also continuous, then the composite function $\zeta = \varphi[f(z)] = F(z)$ is continuous on E.

Let the set E be bounded and closed. Then each function $w = f(z)$ that is continuous on E is bounded on this set, i.e. satisfies the condition $|f(z)| \leqslant C < \infty$, $z \in E$, its modulus attains on E its upper and lower bounds, and, finally, $f(z)$ is *uniformly continuous* on E. The last statement means that for every positive ε there exists a $\delta(\varepsilon) > 0$ such that $|f(z_1) - f(z_2)| < \varepsilon$ for any two points in E for which $|z_1 - z_2| < \delta(\varepsilon)$. All these properties follow from appropriate theorems for functions of two real variables continuous on bounded closed sets, although they can easily be proved directly by repeating almost without change the proofs from calculus courses.

In defining continuity we assumed that $f(z_0) \neq \infty$. When studying mappings by means of analytic functions it is expedient to lift this restriction and assume the function to be continuous at point z_0 where $f(z_0) = \infty$ if

$$\lim_{z \to z_0,\ z \in E} f(z) = \infty.$$

We will call such functions *continuous in an extended sense*. Such functions do not have the above-mentioned properties.

Example. The function $f(z) = 1/z$ for $z \neq 0$ and $z \neq \infty$, which turns zero at point $z = \infty$ and infinity at $z = 0$, is continuous in the extended sense in the extended plane. Indeed, for this function

$$\lim_{z \to \infty} f(z) = 0 = f(\infty), \quad \lim_{z \to 0} f(z) = \infty = f(0).$$

2.4
A CONTINUOUS CURVE

The notion of a continuous curve generalizes the pictorial idea of a curve as the trajectory of a moving particle. In relation to the function $z = \lambda(t)$ of a real variable (parameter) t that is continuous in the closed interval $[\alpha, \beta]$ we say that it defines a *continuous curve* (also a line or arc); the values of the function are called *points on the curve*, and the equation $z = \lambda(t)$ is called the *equation of the curve* (in parametric form). For each curve we may fix one of the two directions in which the curve may be traversed, which corresponds to the parameter either increasing or decreasing. In the first case

$\lambda(\alpha)$ is the beginning and $\lambda(\beta)$ the end of the curve, and in the second their role is interchanged. A curve whose beginning and end coincide is said to be *closed.*

If one point on the curve, z, corresponds to more than one value of parameter t, at least one of which is different from α and β, then such a point is called a *multiple point.* A curve having no multiple point is known as a *simple arc (curve)*, or *Jordan arc.*

Two continuous curves $z = \lambda(t)$ $(\alpha \leqslant t \leqslant \beta)$ and $z = \mu(\tau)$ $(\gamma \leqslant \tau \leqslant \delta)$ are considered identical if and only if there exists a continuous and monotone on the closed interval $[\alpha, \beta]$ function $\tau = \varphi(t)$ such that $\varphi(\alpha) = \gamma$, $\varphi(\beta) = \delta$ (or $\varphi(\alpha) = \delta$, $\varphi(\beta) = \gamma$), and $z = \mu[\varphi(t)] = \lambda(t)$ $(\alpha \leqslant t \leqslant \beta)$.

It can easily be shown that the set of all points on a continuous curve is closed (the proof is left to the reader).

Examples of curves. (a) The equation $z = t$ $(-1 \leqslant t \leqslant 1)$ defines a curve depicted by a segment of the real axis, $-1 \leqslant x \leqslant 1$. For the direction corresponding to the parameter increasing, point -1 is the beginning and point 1 the end of the curve and it is open. The curve has no multiple points and hence it is a Jordan arc.

(b) $z = \cos t$ $(0 \leqslant t \leqslant \pi)$ is a curve identical to the previous one; here the previous direction corresponds to the parameter decreasing.

(c) $z = \cos t$ $(0 \leqslant t \leqslant 2\pi)$. This curve is depicted by the same segment of the real axis, $-1 \leqslant x \leqslant 1$, but it is not identical to the previous curve. Indeed, this is a closed curve since $t = 0$ and $t = 2\pi$ have corresponding to them the same point $z = 1$. Moreover, here two different values of the parameter, t and $2\pi - t$ $(0 < t < 2\pi)$, have corresponding to them a single point $z = \cos t$; hence the curve has multiple points and is not a Jordan arc.

The difference in the continuous curves (a), (b), and (c) lies in the fact that in the first two curves the point z moves along the segment $[-1, 1]$ when the parameter takes on all its values, whereas in the third curve point z moves along the same segment twice.

(d) Let $\Delta_1, \Delta_2, \ldots, \Delta_n$ be rectilinear segments oriented in the plane in such a way that the end of each segment Δ_j $(j = 1, 2, \ldots, n - 1)$ coincides with the beginning of the next segment Δ_{j+1}. Denoting by a_j the complex number depicted by the vector Δ_j and by z_0 the beginning of segment Δ_1, we obtain on the segment $0 \leqslant t \leqslant n$ a simplest continuous function defining a curve depicted by the collection of the given segments:

$$z = z_0 + a_1 + \ldots + a_{j-1} + a_j(t - j + 1)$$

$$(j - 1 \leqslant t \leqslant j,\ j = 1, 2, \ldots, n).$$

This curve is a broken line, and the segments Δ_j are its elements. The broken line is closed or open depending on whether the end of

segment Δ_n coincides with the beginning of segment Δ_1. It is a Jordan arc only if it has no self-intersections, i.e. provided that a common point, and only one such point, have two adjacent segments Δ_j and Δ_{j+1}.

Let us now show that *if any two points z_0 and z' of an open set E can be linked by a continuous curve L contained in E, they can be linked by a broken line contained in E, from which it follows that E is a domain* (see Sec. 1.5).

Indeed, let $z = \lambda(t)$ $(\alpha \leqslant t \leqslant \beta)$ be the equation for L, and $z_0 = \lambda(\alpha)$ and $z' = \lambda(\beta)$. Denote the distance between L and the boundary Γ of E by δ $(\delta > 0)$. Using the fact that $\lambda(t)$ is uniformly continuous in $[\alpha, \beta]$, we divide this segment by points $t_0 = \alpha < t_1 < t_2 < \ldots < t_n = \beta$ into segments so small that $|\lambda(t_{j+1}) - \lambda(t_j)| < \delta$ $(j = 0, 1, \ldots, n-1)$ and link each pair of adjacent points of the curve, $z_j = \lambda(t_j)$ and $z_{j+1} = \lambda(t_{j+1})$, by a chord, Δ_j. Obviously, all chords $\Delta_0, \Delta_1, \ldots, \Delta_{n-1}$ are contained in E and constitute a broken line Λ inscribed in L and connecting z_0 with z'. Hence E is a domain.

At this point we give without proof two theorems concerning simple closed curves and mappings (proofs are given in topology courses).

The Jordan curve theorem. *A simple closed curve (Jordan curve) Γ separates the entire plane into two distinct domains G_1 and G_2 whose common boundary it is. One of the domains (the inner) is bounded while the other (the outer) is not bounded.*

The simplest illustration to the Jordan curve theorem is the inside $(|z - z_0| < \rho)$ and outside $(|z - z_0| > \rho)$ of a circle Γ: $|z - z_0| = \rho$ (in parametric form $z = z_0 + \rho(\cos t + i \sin t)$, $0 \leqslant t \leqslant 2\pi$).

Assume G is an arbitrary domain. If for any Jordan curve γ belonging to G the inner domain also belongs to G, then G is said to be *simply connected* (with respect to the finite plane). An example of a simply connected domain is the interior of a circle. On the other hand, the exterior of a circle and an annulus are not simply connected domains, since for each we can point out a circle that belongs to the domain but whose interior as a whole does not. For conformal mapping theory the notion of a simply connected domain is generalized. Namely, the domain G of the extended plane is said to be *simply connected* (with respect to the extended plane) if for any Jordan curve γ belonging to G the exterior or interior of γ belongs to G, too. All other domains are said to be *multiply connected*. It stands to reason that a domain that is simply connected with respect to the finite plane is simply connected with respect to the extended plane. However, the inverse is not so, generally speaking. For instance, the exterior of a circle to which in the extended plane there belongs the point at infinity is simply connected with respect to the extended plane but is not with respect to the finite plane. On the other hand,

an annulus is not simply connected with respect to both finite and extended planes, i.e. it is multiply connected.

The theorem of one-to-one and continuous mappings. *Let G be a domain in the extended plane and $w = f(z)$ a function that is continuous in the extended sense on G and that maps G in a one-to-one manner onto a set D. Then D is also a domain and the function $z = f^{-1}(w)$, which is the inverse of $f(z)$, is continuous in the extended sense on D. If under the same assumptions $f(z)$ is defined on the boundary Γ of G and in a way such that it is continuous in the extended sense on the closed domain $\bar{G}$, then it maps Γ onto the boundary Δ of D; in other words, the boundary of the image of G coincides with the image of the boundary of G.*

Under an additional assumption that $f(z)$ is analytic in domain G this theorem will be studied in Secs. 10.1, 10.4, and 10.7.

In conclusion of this section we will generalize the notion of a continuous curve. Let $z = \lambda(t)$ be a continuous function in the extended sense in the closed interval $[\alpha, \beta]$ (whose left or right end may lie at infinity). We will say that this function defines a *generalized continuous curve* in the extended plane. If $z = \lambda(t)$ does not become infinite in any point in the interval $[\alpha, \beta]$, the generalized curve does not pass through the point at infinity. The notions of the end points of a curve, the closed curve, a multiple point, and a Jordan arc are naturally generalized to the case of a generalized continuous curve. Examples of generalized Jordan arcs are straight line $z = (\alpha t + \beta) + i(\gamma t + \delta)$ $(-\infty \leqslant t \leqslant +\infty)$, where $\alpha^2 + \gamma^2 \neq 0$ and $z = \infty$ at $t = \pm\infty$, and the parabola $z = (\alpha t^2 + \beta t + \gamma) + i(\delta t + \varepsilon)$, where $\alpha \neq 0$ and $\delta \neq 0$ and $z = \infty$ at $t = \pm\infty$. The hyperbola $z = \frac{a(1+t^2)+2ibt}{1-t^2}$ $(-\infty \leqslant t \leqslant \pm\infty)$, where $a \neq 0$ and $b \neq 0$, is a generalized continuous curve but not a Jordan arc since point $z = \infty$ is the multiple point of the curve (it corresponds to two different values of t: ± 1).

2.5
THE DERIVATIVE AND THE DIFFERENTIAL

Let $f(z)$ be a function of a complex variable, defined and single-valued on a set E, and let z_0 be a point in the set that is a limit point for this set. We make up the expression $\frac{f(z)-f(z_0)}{z-z_0}$. It is obviously a function of z defined for all points in E except z_0. If the limit

$$\lim_{z\to z_0,\ z\in E} \frac{f(z)-f(z_0)}{z-z_0}$$

exists, it is called the *derivative of the function $f(z)$ in E at the point z_0* and is denoted by $f'_E(z_0)$ or, in shorter form, $f'(z_0)$. The function

$f(z)$ that has a derivative is called *differentiable* or *monogenic in E at point* z_0.

In the particular case, where E is a segment of the real axis (finite or infinite), $f(z)$ is a function of a real variable $z = x$ that admits, generally speaking, complex values $f(z) = f(x) = \varphi(x) + i\psi(x)$. If $\psi(x) \neq 0$, i.e. $f(z)$ is complex-valued, then, writing $\frac{f(x)-f(x_0)}{x-x_0}$ in the form $\frac{\varphi(x)-\varphi(x_0)}{x-x_0} + i\,\frac{\psi(x)-\psi(x_0)}{x-x_0}$, we conclude, in view of Sec. 2.2, that $f'(x_0)$ exists if and only if there exist $\varphi'(x_0)$ and $\psi'(x_0)$ with $f'(x_0) = \varphi'(x_0) + i\psi'(x_0)$. For example, if $f(x) = a\cos x + ib\sin x$, then $f'(x) = -a\sin x + ib\cos x$.

Denoting the difference $f(z) - f(z_0)$ by $\Delta f(z)$ (the increment of the function) and $z - z_0$ by Δz (the increment of the independent variable), we write the differentiability condition as

$$\frac{\Delta f(z)}{\Delta z} = f'(z_0) + \varepsilon(z_0, \Delta z),$$

where $\varepsilon(z_0, \Delta z) \to 0$ as $\Delta z \to 0$ $(z \in E)$. This implies that the increment of the function can be written as

$$\Delta f(z) = A\cdot\Delta z + \varepsilon(z_0, \Delta z)\,\Delta z \quad (A = f'(z_0)) \tag{2.1}$$

with A independent of Δz and ε approaching zero as $\Delta z \to 0$. Conversely, a function whose increment can be expressed in form (2.1) with the same conditions imposed on A and $\varepsilon(z_0, \Delta z)$ is differentiable, and its derivative is equal to A. Indeed, from (2.1) it follows that

$$\lim_{\Delta z\to 0} \frac{\Delta f(z)}{\Delta z} \quad (\Delta z = z - z_0, \quad z \in E)$$

exists and is equal to A. Hence, the fact that the increment of a function can be expressed by (2.1) with an A that does not depend on Δz and an ε that tends to zero together with Δz is a necessary and sufficient condition for the differentiability of the function. Note that (2.1) directly implies that a function differentiable at $z_0 \in E$ is continuous at this point (on E).

If we denote Δz by dz (the differential of the independent variable) and $A\cdot\Delta z = f'_E(z_0)\,dz$ by $df(z) = d_E f(z)$ (the differential of the function), we arrive at a relation that expresses the derivative in terms of differentials:

$$f'_E(z_0) = \frac{d_E f(z)}{dz}.$$

Let us take an example to see what role the set E plays in determining whether a function is differentiable. (We recall that E is the set on which the derivative of the function is taken.) First we assume that E is the real axis and $f(z) = f(x) = x$. Then the derivative $f'_E(x)$ exists for every $x \in E$ and is equal to unity, i.e. the function

is differentiable everywhere on E. We continue $f(x)$ into the entire complex plane E_1, still assuming that $f(z) = x$. Obviously this function is continuous at every z and coincides with the initial function when $z \in E$ (i.e. when $y = 0$). But

$$\frac{f(z)-f(z_0)}{z-z_0}=\frac{(x-x_0)}{(x-x_0)+i(y-y_0)}$$

has no limit as $z \to z_0$ (z_0 is any point on the plane) since at $x = x_0$ and $y \neq y_0$ it is zero and at $x \neq x_0$ and $y = y_0$ it is unity. Therefore the function $f(z) = x$ is not differentiable in the plane at any point.

2.6 RULES OF DIFFERENTIATION

From the definition of a derivative and the properties of limits of functions of a complex variable it follows that the main rules of differentiation known from the calculus of differentials can be extended to include the derivatives on the set of functions of a complex variable.

The rules are as follows.

1. If $f(z) \equiv c$, then $\frac{df(z)}{dz}=0$

2. $\frac{d[cf(z)]}{dz}=c\frac{df(z)}{dz}$.

3. $\frac{dz}{dz}=1$.

4. $\frac{d}{dz}[f_1(z)+f_2(z)+\ldots+f_n(z)]=\frac{df_1(z)}{dz}+\frac{df_2(z)}{dz}+\ldots+\frac{df_n(z)}{dz}$.

5. $\frac{d}{dz}[f_1(z)f_2(z)\ldots f_n(z)]=f_2(z)f_3(z)\ldots f_n(z)\frac{df_1(z)}{dz}$

$+f_1(z)f_3(z)\ldots f_n(z)\frac{df_2(z)}{dz}+\ldots+f_1(z)f_2(z)\ldots f_{n-1}(z)\frac{df_n(z)}{dz}$.

6. $\frac{d}{dz}[f(z)]^n=n[f(z)]^{n-1}f'(z)$.

6′. $\frac{d}{dz}(z^n)=nz^{n-1}$.

7. $\frac{d}{dz}(a_0+a_1z+\ldots+a_nz^n)=a_1+2a_2z+\ldots+na_nz^{n-1}$.

8. $\frac{d}{dz}\left[\frac{f_1(z)}{f_2(z)}\right]=\frac{f_2(z)\frac{df_1(z)}{dz}-f_1(z)\frac{df_2(z)}{dz}}{[f_2(z)]^2}$.

Here all the functions $f(z)$, $f_1(z)$, $f_2(z)$, ... are assumed to be differentiable at point z of E. In Rule 8 it is required that $f_2(z)$ be nonzero.

9. Differentiation of composite functions. Assume that the function $w = f(z)$ is differentiable at point $z_0 \in E$ and consider a function $Z = \varphi(w)$ defined on set F of the values of the first function and differentiable at point $w_0 = f(z_0)$ in F. Then the composite function $Z = \varphi[f(z)]$ is differentiable at point z_0 in E, and

$$\frac{d_E \varphi[f(z)]}{dz} = \frac{d_F \varphi(w)}{dw} \frac{d_E f(z)}{dz}.$$

10. Differentiation of inverse functions. Let the function $w = f(z)$ establish a one-to-one correspondence between the points of two sets, E and F, and let the inverse function $z = \varphi(w)$ be continuous on F. Then, if $f(z)$ is differentiable at point $z_0 \in E$ and $f'_E(z_0) \neq 0$, the inverse function $z = \varphi(w)$ is differentiable at point $w_0 = f(z_0) \in \in F$ and

$$\varphi'_F(w_0) = \frac{1}{f'_E(z_0)}.$$

Indeed, $z \neq z_0$ and $w \neq w_0$ due to the fact that the mapping $w = f(z)$ is one-to-one. For this reason we can write

$$\frac{\varphi(w) - \varphi(w_0)}{w - w_0} = \frac{z - z_0}{w - w_0} = \frac{1}{\dfrac{w - w_0}{z - z_0}}.$$

Since $z = \varphi(w)$ tends to $z_0 = \varphi(w_0)$ as $w \to w_0$, we can write

$$\lim_{w \to w_0} \frac{\varphi(w) - \varphi(w_0)}{w - w_0} = \frac{1}{\lim\limits_{z \to z_0} \dfrac{w - w_0}{z - z_0}} = \frac{1}{\lim\limits_{z \to z_0} \dfrac{f(z) - f(z_0)}{z - z_0}} = \frac{1}{f'_E(z_0)},$$

which is what we set out to prove.

2.7
NECESSARY AND SUFFICIENT CONDITIONS FOR DIFFERENTIABILITY AT AN INNER POINT OF A DOMAIN

We will mainly consider functions defined on a certain domain $E = G$, and therefore instead of $f'_E(z)$ or $d_E f(z)/dz$ we will simply write $f'(z)$ or $df(z)/dz$.

We recall that a function of two real variables $u(x, y)$ is called differentiable at point (x_0, y_0) in a domain of its definition if the following relationship is true:

$$u(x, y) - u(x_0, y_0) = A(x_0, y_0)(x - x_0) + B(x_0, y_0)(y - y_0) + \varepsilon_1(x, y; x_0, y_0)(x - x_0) + \varepsilon_2(x, y; x_0, y_0)(y - y_0),$$

where

$$\lim_{\substack{x \to x_0 \\ y \to y_0}} \varepsilon_1(x, y; x_0, y_0) = \lim_{\substack{x \to x_0 \\ y \to y_0}} \varepsilon_2(x, y; x_0, y_0) = 0.$$

The coefficients $A(x_0, y_0)$ and $B(x_0, y_0)$ on the right-hand side of the above relationship are the partial derivatives of $u(x, y)$:

$$A(x_0, y_0) = \left. \frac{\partial u(x, y)}{\partial x} \right|_{\substack{x = x_0 \\ y = y_0}}, \quad B(x_0, y_0) = \left. \frac{\partial u(x, y)}{\partial y} \right|_{\substack{x = x_0 \\ y = y_0}}.$$

Let us prove the following important

Theorem. *For a function $f(z) = u(x, y) + iv(x, y)$ defined on a domain G to be differentiable at a point $z \in G$ as a function of a complex variable, it is necessary and sufficient that the functions $u(x, y)$ and $v(x, y)$ be differentiable at the same point (as functions of two real variables) and that, moreover,*

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}. \tag{2.2}$$

If the conditions of the theorem are fulfilled, the derivative $f'(z)$ may be found in one of the following forms:

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y} = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} = \frac{\partial v}{\partial y} + i\frac{\partial v}{\partial x}. \tag{2.3}$$

Conditions (2.2) have their main usage in the theory of analytic functions and in the applications of this theory to the problems of mechanics and physics. They are known as the *Cauchy-Riemann differential equations.*

We note in passing that this name, generally accepted in academic literature, is unjustified historically since conditions (2.2) were studied in the eighteenth century by Jean Le Rond D'Alembert and especially by Leonhard Euler in the applications of the functions of a complex variable to hydrodynamics (D'Alembert and Euler) and to cartography and integral calculus (Euler). Therefore it would be more correct to change the name and call conditions (2.2) the *D'Alembert-Euler equations.*

We can write the D'Alembert-Euler equations (2.2) in a somewhat different form if we use the so-called formal derivatives. Let $f(z) = u(x, y) + iv(x, y)$ be differentiable in x and y. We introduce new variables ξ and η through the formulas $x = \alpha\xi + \beta\eta$ and $y = \gamma\xi + \delta\eta$, where α, β, γ, and δ are constants. We now have

$$\frac{\partial f}{\partial \xi} = \alpha\frac{\partial f}{\partial x} + \gamma\frac{\partial f}{\partial y}, \quad \frac{\partial f}{\partial \eta} = \beta\frac{\partial f}{\partial x} + \delta\frac{\partial f}{\partial y}.$$

In deriving these two formulas we assumed that α, β, γ, and δ are real numbers and ξ and η are real variables. The next step is to formally extend these relationships to the case of complex variables;

namely, we assume that

$$\alpha=\beta=\frac{1}{2}, \quad \gamma=-\frac{i}{2}, \quad \delta=\frac{i}{2}.$$

Then

$$x=\frac{1}{2}(\xi+\eta), \quad y=\frac{1}{2i}(\xi-\eta),$$

whence $\xi=x+iy=z$ and $\eta=x-iy=\bar{z}$. We obtain

$$\frac{\partial f}{\partial z}=\frac{1}{2}\left(\frac{\partial f}{\partial x}-i\frac{\partial f}{\partial y}\right)=\frac{1}{2}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)+\frac{i}{2}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right),$$

$$\frac{\partial f}{\partial \bar{z}}=\frac{1}{2}\left(\frac{\partial f}{\partial x}+i\frac{\partial f}{\partial y}\right)=\frac{1}{2}\left(\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}\right)+\frac{i}{2}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right).$$

These are the *formal derivatives.* We see that condition (2.2) can be rewritten as

$$\frac{\partial f}{\partial \bar{z}}=0.$$

If this condition is met, we have

$$\frac{\partial f}{\partial z}=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}=f'(z).$$

We now turn to the proof of the theorem. First we show that the hypothesis of the theorem is the necessary condition for the differentiability of $f(z)$.

Indeed, if $f(z)$ is differentiable at point z in domain G, then

$$\Delta f(z)=f'(z)\,\Delta z+\varepsilon\,\Delta z, \tag{2.4}$$

where

$$\Delta z=z_1-z=(x_1-x)+i(y_1-y)=\Delta x+i\,\Delta y,$$

$$\Delta f(z)=f(z_1)-f(z)=[u(x_1,y_1)-u(x,y)]$$
$$+\,i\,[v(x_1,y_1)-v(x,y)]=\Delta u+i\,\Delta v,$$

$$f'(z)=a+ib, \quad \varepsilon=\varepsilon_1+i\varepsilon_2,$$

and ε_1 and ε_2 tend to zero as Δx and Δy tend to zero simultaneously.

Separating the real and imaginary parts in (2.4), we obtain

$$\Delta u=a\,\Delta x-b\,\Delta y+\varepsilon_1\,\Delta x-\varepsilon_2\,\Delta y,$$
$$\Delta v=b\,\Delta x+a\,\Delta y+\varepsilon_2\,\Delta x-\varepsilon_1\,\Delta y.$$

Whence due to the fact that $\lim\limits_{\Delta x,\,\Delta y\to 0}\varepsilon_1=\lim\limits_{\Delta x,\,\Delta y\to 0}\varepsilon_2=0$ we see that

(1) the functions $u(x,y)$ and $v(x,y)$ of two real variables x and y are differentiable at point (x,y);

(2) their partial derivatives at this point are

$$\frac{\partial u}{\partial x}=a, \quad \frac{\partial u}{\partial y}=-b, \quad \frac{\partial v}{\partial x}=b, \quad \frac{\partial v}{\partial y}=a$$

and, therefore, satisfy the condition

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}.$$

Finally, for $f'(z)$ we have

$$f'(z)=a+ib=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}=\frac{\partial v}{\partial y}-i\frac{\partial u}{\partial y}=\frac{\partial u}{\partial x}-i\frac{\partial u}{\partial y}=\frac{\partial v}{\partial y}+i\frac{\partial v}{\partial x}.$$

Therefore we have proved the necessity of the theorem's hypothesis for the differentiability of a function.

Next we show that the hypothesis is the sufficient condition for the differentiability of $f(z)$. Let us assume that the hypothesis is true. Then

$$\left.\begin{aligned}\Delta u&=\frac{\partial u}{\partial x}\Delta x+\frac{\partial u}{\partial y}\Delta y+\alpha_1\Delta x+\alpha_2\Delta y,\\ \Delta v&=\frac{\partial v}{\partial x}\Delta x+\frac{\partial v}{\partial y}\Delta y+\beta_1\Delta x+\beta_2\Delta y,\end{aligned}\right\} \tag{2.5}$$

where α_1, α_2, β_1, and β_2 tend to zero as Δx and Δy tend to zero simultaneously. Moreover,

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}=a, \quad -\frac{\partial u}{\partial y}=\frac{\partial v}{\partial x}=b. \tag{2.6}$$

Hence

$$\begin{aligned}\Delta u&=a\,\Delta x-b\,\Delta y+\alpha_1\,\Delta x+\alpha_2\,\Delta y,\\ \Delta v&=b\,\Delta x+a\,\Delta y+\beta_1\,\Delta x+\beta_2\,\Delta y\end{aligned}$$

and

$$\begin{aligned}\Delta f(z)&=\Delta u+i\,\Delta v=a(\Delta x+i\,\Delta y)+ib(\Delta x+i\,\Delta y)+(\alpha_1+i\beta_1)\Delta x\\ &\quad+(\alpha_2+i\beta_2)\Delta y=(a+ib)\Delta z+\left[(\alpha_1+i\beta_1)\frac{\Delta x}{\Delta z}+(\alpha_2+i\beta_2)\frac{\Delta y}{\Delta z}\right]\Delta z\\ &=A\,\Delta z+\varepsilon\,\Delta z.\end{aligned} \tag{2.7}$$

Since

$$\begin{aligned}|\varepsilon|&=\left|(\alpha_1+i\beta_1)\frac{\Delta x}{\Delta z}+(\alpha_2+i\beta_2)\frac{\Delta y}{\Delta z}\right|\\ &\leqslant|\alpha_1+i\beta_1|\left|\frac{\Delta x}{\Delta z}\right|+|\alpha_2+i\beta_2|\left|\frac{\Delta y}{\Delta z}\right|\\ &\leqslant|\alpha_1+i\beta_1|+|\alpha_2+i\beta_2|\leqslant|\alpha_1|+|\beta_1|+|\alpha_2|+|\beta_2|,\end{aligned}$$

we see that ε together with α_1, α_2, β_1, and β_2 tend to zero as $\Delta z = \Delta x + i\,\Delta y \to 0$. From this and (2.7) it follows that the function is differentiable and that its derivative $f'(z)$ is equal to A:

$$f'(z)=A=a+ib=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}=\dots,$$

which completes the proof.

From the course in calculus we know that for the functions $u(x, y)$ and $v(x, y)$ to be differentiable it is sufficient that the partial derivatives $\partial u/\partial x$, $\partial u/\partial y$, $\partial v/\partial x$, and $\partial v/\partial y$ exist and are continuous. Therefore for $f(z) = u + iv$ to be differentiable it is sufficient that these partial derivatives exist, are continuous, and satisfy Eq. (2.2).

A function $f(z)$ that is differentiable at each point of a domain G is said to be differentiable in G and is also called *holomorphic* or *analytic* or *regular*. The name holomorphic (from the Greek *holos* for whole and *morphē* for form) was introduced by the pupils of Augustin Louis Cauchy, C.A.A. Briot and J.C. Bouquet. "By this name", they wrote, "we wish to show that it [the holomorphic function] is similar to entire ['whole'] functions [a rational entire function is a polynomial] with respect to the properties on the entire z-plane." The meaning of the term analytic, which was used by Joseph Louis Lagrange and later by Karl Theodor Wilhelm Weierstrass and is today widely used, was explained in the Introduction. Its applicability to functions of a complex variable that are differentiable in a certain domain will be justified in our subsequent exposition, where we will show that such a function can be expressed inside a neighborhood of every point of that domain by a converging power series. For the time being we will use the term "analytic function" as a synonym for "a function of a complex variable differentiable in a given domain".

As an example, we take the function $f(z) = e^x(\cos y + i \sin y)$ defined in the entire plane. Here

$$u = e^x \cos y, \quad v = e^x \sin y$$

are differentiable functions:

$$\frac{\partial u}{\partial x} = e^x \cos y = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -e^x \sin y = -\frac{\partial v}{\partial x}.$$

Hence, condition (2.2) is satisfied and $f(z)$ is analytic in the entire plane. Its derivative is

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = e^x \cos y + ie^x \sin y = f(z).$$

In the example at the end of Sec. 2.5, $f(z) = x$, $u = x$, $v = 0$, $\partial u/\partial x = 1$, $\partial u/\partial y = 0$, $\partial v/\partial x = 0$, and $\partial v/\partial y = 0$ and the D'Alembert-Euler equations do not work: $\partial u/\partial x \neq \partial v/\partial y$. We see that this function is not differentiable (in the plane).

In many cases it is necessary to have criteria for differentiability at a point $z \neq 0$ of a function of a complex variable, $f(z) = u + iv$, expressed in terms of polar coordinates $|z| = r$ and $\operatorname{Arg} z = \Phi$. These (necessary and sufficient) conditions are

(1') u and v are differentiable functions of r and Φ;

(2′) their partial derivatives are related thus:

$$\frac{\partial u}{\partial r}=\frac{1}{r}\frac{\partial v}{\partial \Phi}, \quad \frac{\partial v}{\partial r}=-\frac{1}{r}\frac{\partial u}{\partial \Phi}. \tag{2.8}$$

To prove this it is sufficient to show that u and v are differentiable functions of r and Φ ($r \neq 0$) if and only if they are differentiable functions of x and y and that under such conditions Eqs. (2.8) are equivalent to Eqs. (2.2). But the first requirement follows from the known fact of calculus that a differentiable function (e.g. $u = u(x, y)$) of differentiable functions (e.g. $x = r\cos\Phi$ and $y = r\sin\Phi$) is differentiable with respect to the new variables (r and Φ). The second requirement can be verified directly. For example, if condition (1′) is satisfied and so is (2.2), then

$$\left.\begin{aligned}\frac{\partial u}{\partial r}&=\frac{\partial u}{\partial x}\cos\Phi+\frac{\partial u}{\partial y}\sin\Phi=\frac{\partial v}{\partial y}\cos\Phi-\frac{\partial v}{\partial x}\sin\Phi=\frac{1}{r}\frac{\partial v}{\partial \Phi},\\ \frac{\partial v}{\partial r}&=\frac{\partial v}{\partial x}\cos\Phi+\frac{\partial v}{\partial y}\sin\Phi=-\frac{\partial u}{\partial y}\cos\Phi+\frac{\partial u}{\partial x}\sin\Phi=-\frac{1}{r}\frac{\partial u}{\partial \Phi}.\end{aligned}\right\} \tag{2.9}$$

The reader can easily perform the opposite transformation from (2.8) to (2.2).

The necessity for (2.8) is most easily perceived by the following method. Through the point $z_0 = r_0(\cos\Phi_0 + i\sin\Phi_0)$ let us draw a ray emerging from the origin of coordinates and a circle with the center at the origin. If z approaches z_0 along the ray, then for it $z = r(\cos\Phi_0 + i\sin\Phi_0)$ and $r \to r_0$. Whence

$$f'(z_0)=\lim_{r\to r_0}\frac{f(z)-f(z_0)}{(r-r_0)(\cos\Phi_0+i\sin\Phi_0)}=\frac{r_0}{z_0}\lim_{r\to r_0}\frac{f(z)-f(z_0)}{r-r_0}=\frac{r_0}{z_0}\left.\frac{\partial f}{\partial r}\right|_{z_0}.$$

On the other hand, we will arrive at the same derivative if we send z to z_0 along the circle. Then $z = r_0(\cos\Phi + i\sin\Phi)$ and $\Phi \to \Phi_0$; hence

$$\begin{aligned}f'(z_0)&=\lim_{\Phi\to\Phi_0}\left[\frac{f(z)-f(z_0)}{\Phi-\Phi_0}\Big/\frac{z-z_0}{\Phi-\Phi_0}\right]=\left.\frac{\partial f}{\partial \Phi}\right|_{z_0}\Big/\left.\frac{\partial z}{\partial \Phi}\right|_{z_0}\\ &=\left.\frac{\partial f}{\partial \Phi}\right|_{z_0}\Big/[r_0(-\sin\Phi_0+i\cos\Phi_0)]=\frac{1}{iz_0}\left.\frac{\partial f}{\partial \Phi}\right|_{z_0}.\end{aligned}$$

Comparing the two results and dropping the "0" to simplify notation, we obtain

$$\frac{r}{z}\frac{\partial f}{\partial r}=\frac{1}{iz}\frac{\partial f}{\partial \Phi}, \quad \text{i.e.} \quad \frac{1}{r}\frac{\partial f}{\partial \Phi}=i\frac{\partial f}{\partial r}.$$

After substituting $\frac{\partial f}{\partial \Phi}=\frac{\partial u}{\partial \Phi}+i\frac{\partial v}{\partial \Phi}$, $\frac{\partial f}{\partial r}=\frac{\partial u}{\partial r}+i\frac{\partial v}{\partial r}$, we arrive at (2.8).

In passing we have found formulas for calculating $f'(z)$ in polar coordinates:

$$f'(z)=\frac{r}{z}\frac{\partial f}{\partial r}=\frac{r}{z}\left(\frac{\partial u}{\partial r}+i\frac{\partial v}{\partial r}\right) \tag{2.10}$$

and

$$f'(z)=\frac{1}{iz}\frac{\partial f}{\partial \Phi}=\frac{1}{z}\left(\frac{\partial v}{\partial \Phi}-i\,\frac{\partial u}{\partial \Phi}\right). \tag{2.11}$$

Let us consider the function

$$z^{\frac{m}{n}}=|\,z\,|_{+}^{\frac{m}{n}}\left(\cos\frac{m\operatorname{Arg} z}{n}+i\sin\frac{m\operatorname{Arg} z}{n}\right)=r_{+}^{\frac{m}{n}}\left(\cos\frac{m\Phi}{n}+i\sin\frac{m\Phi}{n}\right)$$

as an example. Here m is an integer and n a positive integer. The function is defined in domain $G{:}z\neq 0$ and is many-valued if the rational number m/n is not an integer (see Sec. 1.2). The multivalence of $z^{m/n}$ is due to the multivalence of argument Φ. In order to be able to define a derivative of this many-valued function at a point z in G, we take in G a neighborhood of z without the origin of coordinates and fix one of the values of Φ at z. Next we take at all other points z_1 of the same neighborhood those values of the argument, Φ_1, that satisfy the condition $|\,\Phi_1-\Phi\,|<\pi/2$ (Fig. 5). Then in this neighborhood we obtain a single-valued and continuous branch of function $z^{m/n}$, which we denote in the same way as the initial function: $f(z)=z^{m/n}$. Obviously, in this case

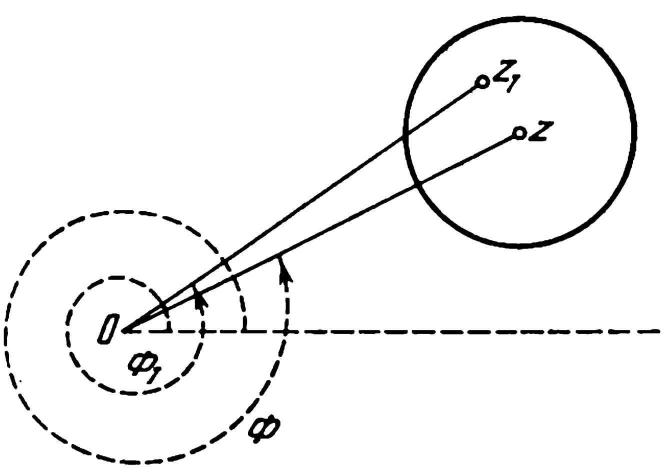

Fig. 5

$$u=r_{+}^{\frac{m}{n}}\cos\frac{m\Phi}{n},\quad v=r_{+}^{\frac{m}{n}}\sin\frac{m\Phi}{n},$$

$$\frac{\partial u}{\partial r}=\frac{m}{n}\,r_{+}^{\frac{m}{n}-1}\cos\frac{m\Phi}{n}=\frac{1}{r}\,\frac{\partial v}{\partial \Phi},$$

$$\frac{\partial v}{\partial r}=\frac{m}{n}\,r_{+}^{\frac{m}{n}-1}\sin\frac{m\Phi}{n}=-\frac{1}{r}\,\frac{\partial u}{\partial \Phi},$$

and hence $f(z)$ is a differentiable function. In view of (2.10) its derivative is

$$f'(z)=\frac{r}{z}\left(\frac{m}{n}\,r_{+}^{\frac{m}{n}-1}\cos\frac{m\Phi}{n}+\frac{m}{n}\,r_{+}^{\frac{m}{n}-1}\sin\frac{m}{n}\Phi\right)$$

$$=\frac{m}{n}\,r_{+}^{\frac{m}{n}}\left(\cos\frac{m\Phi}{n}+\sin\frac{m\Phi}{n}\right)\frac{1}{z}=\frac{m}{n}\,\frac{f(z)}{z}.$$

The reader can see that the rule of differentiation for $z^{m/n}$ formally remains the same as the rule for the function $x^{m/n}$ of a real

variable. It must be borne in mind, however, that our "proof" assumes that $z \neq 0$. This restriction can be lifted only if m/n is a nonnegative integer.

We advise the reader to check whether the function $f(z) = \ln r + i\Phi$ defined in the same domain G is differentiable and whether its derivative is $1/z$. (Here one must also separate single-valued continuous branches.)

2.8 THE GEOMETRIC INTERPRETATION OF THE ARGUMENT OF A DERIVATIVE

We start with a complex-valued function $z = \lambda(t)$ of a real variable t, a function that is defined and continuous in an interval E: $[\alpha, \beta]$ of the real axis. As said in Sec. 2.4, this function defines a continuous curve L. Let us assume that at a point t_0 in $[\alpha, \beta]$ there exists the derivative (in E) $\lambda'(t_0) \neq 0$. We will then see that at point $z_0 = \lambda(t_0)$ of L there exists a tangent T (understood as the limiting position of the secant at point z_0), and the angle between this tangent and the real axis is $\operatorname{Arg} \lambda'(t_0)$.

Indeed, let us pass a secant through the points $z_0 = \lambda(t_0)$ and $z_1 = \lambda(t_1)$ of curve L. It can be assumed that the points do not coincide for all points t_1 that differ from t_0 but are sufficiently close to t_0; otherwise we may find a sequence $\{t_{1n}\} \to t_0$ such that

$$\lambda(t_{1n}) - \lambda(t_0) = 0$$

for all values of n and hence

$$\lambda'(t_0) = \lim_{t_{1n} \to t_0} \frac{\lambda(t_{1n}) - \lambda(t_0)}{t_{1n} - t_0} = 0.$$

Noting that the direction of the secant coincides with that of the vector $(z_1 - z_0)/(t_1 - t_0)$, we conclude that the secant has its limiting position as $t_1 \to t_0$ $(z_1 \to z_0)$ only if the angle between the above-mentioned vector and the real axis, which is $\operatorname{Arg} [(z_1 - z_0)/(t_1 - t_0)]$, attains a limit as $t_1 \to t_0$. But since we have assumed that there exists the limit

$$\lim_{t_1 \to t_0} \frac{z_1 - z_0}{t_1 - t_0} = \lambda'(t_0) \neq 0,$$

there also exists the limit

$$\lim_{t_1 \to t_0} \operatorname{Arg} \frac{z_1 - z_0}{t_1 - t_0} = \operatorname{Arg} \lambda'(t_0),$$

which completes the proof.

Hence, *for a complex-valued function of a real variable, the existence of a nonzero derivative leads to the existence of a tangent to the corresponding curve; the slope of this tangent in relation to the real axis coincides with the argument of the derivative.*

We turn now to the function of a complex variable, $w = f(z)$, defined and continuous on a domain G. We assume that at point $z_0 \in G$ there exists a nonzero derivative $f'(z_0)$. Through point z_0 we draw a curve L: $z = \lambda(t)$ $(\alpha \leqslant t \leqslant \beta,\ \lambda(t_0) = z_0)$; the function $\lambda(t)$ has a nonzero derivative $\lambda'(t_0)$ at this point, and curve L has a tangent at this point with a slope equal to $\operatorname{Arg} \lambda'(t_0)$. By means of the mapping $w = f(z)$ this curve is transformed into curve Λ in the w-plane: $w = f[\lambda(t)] = \mu(t)$ $(\alpha \leqslant t \leqslant \beta,\ \mu(t_0) = f(z_0) = w_0)$. According to the rule for finding the derivative of a composite function, $\mu(t)$ is differentiable at point $t = t_0$ and $\mu'(t_0) = f'(z_0)\lambda'(t_0) \neq 0$, and curve Λ has a tangent at point $w_0 = f(z_0)$. The angle between this tangent and the real axis is

$$\operatorname{Arg} \mu'(t_0) = \operatorname{Arg} [\lambda'(t_0) f'(z_0)] = \operatorname{Arg} \lambda'(t_0) + \operatorname{Arg} f'(z_0).$$

This implies that when we go over from curve L to its image Λ, the slope of the tangent at t_0 changes by

$$\operatorname{Arg} \mu'(t_0) - \operatorname{Arg} \lambda'(t_0) = \operatorname{Arg} f'(z_0),$$

a quantity that depends only on $f'(z_0)$, not on the specific shape of L.

Hence, $\operatorname{Arg} f'(z_0)$ *is equal to the angle of rotation of the tangent to L at point z_0 in the course of transformation to the image Λ and to the point $w_0 = f(z_0)$*. This is the *geometric interpretation* of $\operatorname{Arg} f'(z_0)$. In particular, if $f'(z_0)$ is a real positive number, the vectors of the tangents to L at z_0 and to Λ at w_0 are parallel and point in the same direction. If two curves L_1 and L_2 emerge from point z_0 and have tangents T_1 and T_2 at this point, then the tangents τ_1 and τ_2 of the image curves Λ_1 and Λ_2 at the point $w_0 = f(z_0)$ are obtained from T_1 and T_2 by rotating T_1 and T_2 by the same angle $\operatorname{Arg} f'(z_0)$. Hence, the angle between L_1 and L_2 is equal (in magnitude and in sense of rotation) to that between Λ_1 and Λ_2. Therefore, *in the course of mapping by means of a function $w = f(z)$ that is continuous in the neighborhood of point z_0 and has a nonzero derivative $f'(z_0)$, all the curves in the z-plane that pass through z_0 and have tangents at this point are mapped into curves in the w-plane that pass through $w_0 = f(z_0)$ and have tangents at this point. The angles between these curves are preserved in the mapping.*

A mapping by means of a continuous function that preserves the angle between curves passing through a given point is called *conformal at the point*. If not only the magnitude of the angles but the sense of rotation remain the same (as in the above mapping), we speak of *conformal mapping of the first kind*; but if the sense of rotation changes to opposite (for instance, in the case of mirror reflection with respect to the real axis: $w = \bar{z}$), then we speak of *conformal mapping of the second kind*.

Hence, *mapping by means of a function of a complex variable that is analytic in a domain G is a conformal mapping of the first kind at all points where the derivative is nonzero.* If the mapping is conformal at all points of the domain G, then we speak of the *conformal mapping* of domain G.

A general example of a conformal mapping of the second kind is mappings by means of functions that are conjugate complex to analytic functions: $w = \overline{f(z)}$ (we assume that $f'(z) \neq 0$ holds).

The reader is advised to prove that if the derivative at a point vanishes, the angle may be preserved or may not. (Consider the mappings

$$f_1(z) = r^2 (\cos \Phi + i \sin \Phi)$$

and

$$f_2(z) = r^2 (\cos 2\Phi + i \sin 2\Phi) = z^2$$

at point $z = 0$.)

2.9

THE GEOMETRIC INTERPRETATION OF THE MODULUS OF A DERIVATIVE

Let us now examine the geometric interpretation of $|f'(z_0)|$. To this end we note that

$$|f'(z_0)| = \lim_{z \to z_0} \frac{|f(z) - f(z_0)|}{|z - z_0|}$$

and that $|z - z_0|$ and $|f(z) - f(z_0)|$ are the distances between points z and z_0 in the z-plane and between their images $f(z)$ and $f(z_0)$ in the w-plane, respectively. If we can consider $\frac{|f(z) - f(z_0)|}{|z - z_0|}$ as an expansion of vector $z - z_0$ as a result of mapping by means of the function $w = f(z)$ (this expansion may be greater than or equal to or less than unity), then the *modulus of the derivatie*, $|f'(z_0)|$, *can be considered an expansion at point z_0 as a result of mapping by means of the function $w = f(z)$*. The magnitude of this expansion at point z_0, as follows from the aforementioned, does not depend on the vector $z - z_0$ emerging from this point. However, this expansion is not the expansion of the vector $z - z_0$ but the limit of such an expansion as $z \to z_0$.

2.10

THE LINEAR AND LINEAR-FRACTIONAL FUNCTIONS

As an illustration we consider the *linear-fractional function* $L(z) = (az + b)/(cz + d)$ (where at least one of the numbers c or d is not zero). First let $c = 0$. Then we can write $L(z)$ in the form $L(z) = \alpha z + \beta$ ($\alpha = a/d$ and $\beta = b/d$), which is the *entire linear function*.

It is defined for all values of z and has a derivative $L'(z) = \alpha$ that remains constant and is nonzero if $\alpha \neq 0$. Hence, the function $L(z)$ maps conformally the entire complex z-plane. In this mapping the tangents to every curve in the z-plane rotate through the same angle $\operatorname{Arg} \alpha$ and the expansion at all points turns out to be $|\alpha|$. If $\alpha = 1$, then $\operatorname{Arg} \alpha = 2k\pi$, $|\alpha| = 1$, and both rotation and expansion are absent. Since the mapping in this case takes the form $w = z + \beta$, it reduces to shifting the plane as a whole by the vector β. But if $\alpha \neq 1$ (and $\alpha \neq 0$), then we can present the mapping as $w - \gamma = \alpha (z - \gamma)$, where γ is determined from the equation $\gamma = \alpha\gamma + \beta$. This implies that each vector $z - \gamma$ starting at point γ rotates as a result of mapping through an angle $\operatorname{Arg} \alpha$ and expands $|\alpha|$ times, turning as a result into vector $w - \gamma$ starting at γ, too. This means that the mapping $L(z) = \alpha z + \beta$ at $\alpha \neq 1$ (and $\alpha \neq 0$) is reduced to rotation of the entire plane as a whole about point $\gamma = \beta/(1 - \alpha)$ through angle $\operatorname{Arg} \alpha$ and expansion $|\alpha|$ times with respect to γ. Obviously, this is a similarity transformation with the center at point $\gamma = \beta/(1 - \alpha)$ and the similarity coefficient $|\alpha|$ that is accompanied by a rotation about the same point through an angle $\operatorname{Arg} \alpha$. This is conformal mapping in its simplest form.

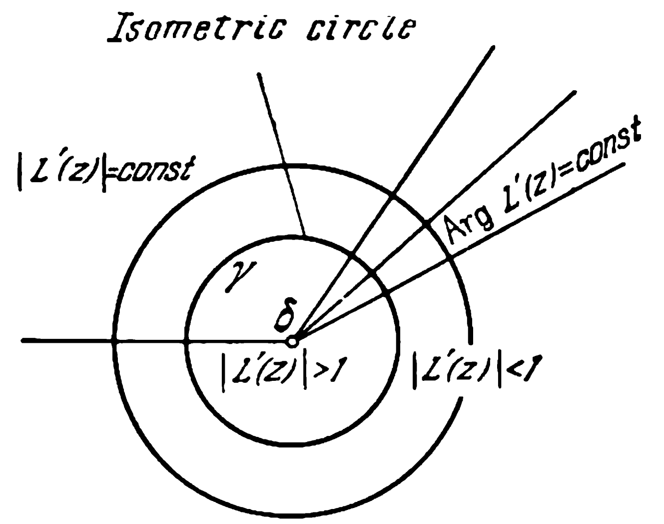

Fig. 6

Now we assume that $c \neq 0$. Then at $z \neq \delta = -d/c$ there exists a derivative

$$L'(z) = \frac{ad - bc}{(cz + d)^2} = \frac{ad - bc}{c^2} \frac{1}{(z - \delta)^2}.$$

If the determinant $ad - bc \neq 0$ (if $ad - bc = 0$, then $a/c = b/d = \lambda$, whence $a = c\lambda$, $b = d\lambda$, and $L(z) = (az + b)/(cz + d) = (c\lambda z + d\lambda)/(cz + d) \equiv \lambda$), then $L'(z) \neq 0$ for all $z \neq \delta$. Hence, the mapping $w = L(z)$ is conformal at all finite points distinct from δ. For this mapping, the tangents to curves passing through an arbitrary point $z \neq \delta$ rotate through an angle

$$\operatorname{Arg} L'(z) = \operatorname{Arg} \frac{ad - bc}{c^2} - 2 \operatorname{Arg} (z - \delta).$$

The angle of rotation of a tangent obviously changes from point to point but has the same value for those points for which $\operatorname{Arg} (z - \delta)$ preserves its value, i.e. for the point of rays starting at δ. The expansion at point z for the given mapping is $|L'(z)| = |(ad - bc)/c^2|/$

$/|z - \delta|^2$ and also changes from point to point. It preserves its values for those points for which $|z - \delta|$ preserves its value, i.e. for the points of circles with centers at δ. In particular, this expansion is equal to unity for each point of the circle γ: $|z - \delta| = |c|^{-1}(|ad - bc|)^{1/2}$ (the *isometric circle of the linear-fractional transformation*), is greater than unity inside γ (tending to ∞ as $z \to \delta$) and is less than unity outside γ (tending to 0 as $z \to \infty$) (see Fig. 6).

2.11
AN ANGLE WITH ITS VERTEX AT THE POINT AT INFINITY

We again assume that $c \neq 0$ and $ad - bc \neq 0$. Then, obviously,

$$\lim_{z \to \delta = -\frac{d}{c}} \frac{az+b}{cz+d} = \infty \text{ and } \lim_{z \to \infty} \frac{az+b}{cz+d} = \frac{a}{c} = \lambda.$$

We supplement the definition of $L(z)$ by assuming that

$$L(\delta) = \infty \quad \text{and} \quad L(\infty) = \lambda.$$

The function $w = L(z)$ is defined now in the extended z-plane, and it maps the finite point δ into the point at infinity in the w-plane and the point at infinity in the z-plane into the finite point λ.

From the equation $w = (az + b)/(cz + d)$ we obtain for the inverse function $z = L^{-1}(w)$ the following expression:

$$z = \frac{-dw+b}{cw-a}.$$

Here we have first assumed that $w \neq \infty$ and $w \neq \lambda$ (then $z \neq \delta$ and $z \neq \infty$). For $w = \infty$ and $w = \lambda$ we have the following values: $L^{-1}(\infty) = \delta$ and $L^{-1}(\lambda) = \infty$.

Therefore, a function that is the inverse of a linear-fractional function is itself linear-fractional. We also can see that a linear-fractional function $w = L(z)$ performs a one-to-one mapping of the extended complex plane into itself and that this mapping is conformal at $z \neq \delta$ and $z \neq \infty$. In order to say that the mapping is conformal at δ and ∞, we must define the notion of angle with the vertex at the point at infinity. Suppose that C_1 and C_2 are two continuous curves that pass through the origin of coordinates at which they form an angle θ. We map the plane into itself by means of the function $\zeta = 1/z$. Then C_1 and C_2 transform into continuous (in the extended sense) curves C_1' and C_2' that pass through the point at infinity. We then speak of curves C_1' and C_2' forming an angle θ at the point at infinity. For instance, the real and imaginary axes form at origin the angle $\pi/2$. The function $\zeta = 1/z$ maps each of these axes into themselves and point $z = 0$ is mapped into point $\zeta = \infty$.

This implies that the real and imaginary axes intersect at infinity at the same angle $\pi/2$.

Generally speaking, *any two curves C'_1 and C'_2 pass through point $z = \infty$ and form an angle θ at that point if the image curves C_1 and C_2 in the course of the mapping $\zeta = 1/z$ form an angle θ at the origin of coordinates.*

We return to the general case of a mapping by means of the function

$$w = L(z) = \frac{az+b}{cz+d} \quad (ad - bc \neq 0, \quad c \neq 0).$$

Suppose two curves C_1 and C_2 form an angle θ with the vertex at point $\delta = -d/c$. Since $L(\delta) = \infty$, the image curves $C'_1 = L(C_1)$ and $C'_2 = L(C_2)$ pass through point $w = \infty$. By definition, the angle between these image curves must be equal to the angle with the vertex at the origin of coordinates between curves C''_1 and C''_2 obtained by means of the mapping $\zeta = 1/w$. Obviously, C''_1 and C''_2 are image curves for C_1 and C_2 as a result of the mapping $\zeta = 1/w = (cz + d)/(az + b)$. But the latter is a linear-fractional transformation and hence is conformal at $z = \delta$ (which it maps into the finite point $\zeta = 0$). For this reason the curves C''_1 and C''_2 form at $\zeta = 0$ an angle θ, i.e. the curves C'_1 and C'_2 form at $w = \infty$ an angle θ. Hence, we have established that the mapping $w = L(z)$ is conformal at point δ, at which the linear-fractional function becomes infinite. This result remains valid, of course, for the inverse function $z = L^{-1}(w)$ at point λ at which it becomes infinite. In other words, the mapping $w = L(z)$ is conformal at the point at infinity, too (which it maps into λ). Summing up, we can state that *each linear-fractional function $w = (az + b)/(cz + d)$ (with $ad - bc \neq 0$ and $c \neq 0$) performs a one-to-one conformal mapping of the extended plane into itself.*

The reader can easily check this statement for all entire linear functions $w = az + b$, the only difference being that in this case the point at infinity is carried into itself.

2.12

QUASICONFORMAL MAPPINGS

Consider a function $f(z) = u(x, y) + iv(x, y)$ that may not be analytic in the general case. Suppose that u and v are differentiable with respect to x and y in a domain G. The mapping $w = f(z)$ can be approximated in the neighborhood of point $z_0 = x_0 + iy_0 \in G$ by the linear mapping

$$w = w_0 + du + i\,dv,$$

or

$$\left.\begin{aligned} u &= u_0 + a\,(x - x_0) + b\,(y - y_0), \\ v &= v_0 + c\,(x - x_0) + d\,(y - y_0), \end{aligned}\right\} \tag{2.12}$$

where

$$a = u'_x\,(x_0,\ y_0), \quad b = u'_y\,(x_0,\ y_0),$$
$$c = v'_x\,(x_0,\ y_0), \quad d = v'_y\,(x_0,\ y_0).$$

Obviously, the image of the point $z = x + iy$ will differ from $w = f\,(z)$ as a result of mapping (2.12) by a quantity of the order of $\varepsilon \mid z - z_0 \mid$, where $\varepsilon \to 0$ as $z \to z_0$. Let point z describe a circle $\mid z - z_0 \mid = \rho$; assuming that $x = x_0 + \rho\cos\theta$ and $y = y_0 + \rho\cos\theta$, $0 \leqslant \theta \leqslant 2\pi$, we find that as a result of mapping (2.12) z describes the following curve:

$$\left.\begin{aligned} u &= u_0 + (a\cos\theta + b\sin\theta)\,\rho, \\ v &= v_0 + (c\cos\theta + d\sin\theta)\,\rho. \end{aligned} \quad (0 \leqslant \theta \leqslant 2\pi)\right\} \tag{2.13}$$

If $a = b = c = d = 0$, this curve is simply point (u_0, v_0), while even if one partial derivative is nonzero and $D = D\,(u,\ v)/D\,(x,\ y)\,|_{z_0} = ad - bc \neq 0$, the curve is an ellipse (the ellipse degenerates into a line segment if $D = 0$). To see how the distance between w and w_0 changes in the process, consider the fraction

$$\begin{aligned} \frac{|\,w - w_0\,|}{\rho} &= |\,(a\cos\theta + b\sin\theta) + i\,(c\cos\theta + d\sin\theta)\,| \\ &= \sqrt{(a\cos\theta + b\sin\theta)^2 + (c\cos\theta + d\sin\theta)^2} \\ &= \sqrt{\frac{1}{2}A + \frac{1}{2}\sqrt{A^2 - 4D^2}\,\sin\,(2\theta + \theta_0)}, \end{aligned}$$

where

$$A = a^2 + b^2 + c^2 + d^2$$

and

$$\tan\theta_0 = \frac{a^2 + c^2 - b^2 - d^2}{2\,(ab + cd)}.$$

Whence

$$\max_{|z - z_0| = \rho} |\,w - w_0\,| = \rho\sqrt{\frac{1}{2}A + \frac{1}{2}\sqrt{A^2 - 4D^2}},$$

$$\min_{|z - z_0| = \rho} |\,w - w_0\,| = \rho\sqrt{\frac{1}{2}A - \frac{1}{2}\sqrt{A^2 - 4D^2}}$$

and

$$1 \leqslant p\,(z_0) = \frac{\max\,|\,w - w_0\,|}{\min\,|\,w - w_0\,|} = \sqrt{\frac{A + \sqrt{A^2 - 4D^2}}{A - \sqrt{A^2 - 4D^2}}} = \frac{A + \sqrt{A^2 - 4D^2}}{2D}. \tag{2.14}$$

From (2.14) it follows that the curve (2.13) is a circle with its center at point (u_0, v_0) if and only if

$$A^2 - 4D^2 = 0, \quad \text{i.e.} \quad a^2 + b^2 + c^2 + d^2 = 2\,|\,ad - bc\,|.$$

We will distinguish between the case where $D > 0$ (point w according to Eq. (2.13) describes a circle in the same direction as point z describes a circle $|\,z - z_0\,| = \rho$) and the case where $D < 0$ (here the directions are opposite to each other). In the first case

$$(a - d)^2 + (b + c)^2 = 0,$$

i.e.

$$a = d \quad \text{and} \quad b = -c,$$

which are the D'Alembert-Euler equations for $f(z)$. If $D < 0$,

$$(a + d)^2 + (b - c)^2 = 0,$$

i.e.

$$a = -d \quad \text{and} \quad b = c,$$

which are the D'Alembert-Euler equations for $\overline{f(z)}$, the conjugate complex of $f(z)$.

Let us suppose that $D = D(u, v)/D(x, y) > 0$ holds for all points in domain G. Then we can see that the analytic function $f(z) = u + iv$ is fully characterized if we require that every circle $|\,z - z_0\,| = \rho$ be mapped by $f(z)$ into the circle $|\,w - w_0\,| = r$ (to within higher-order infinitesimals in ρ). In the general case the image of a circle $|\,z - z_0\,| = \rho$ is an ellipse (2.13) with the ratio $p(z_0)$ of the major semi-axis to the minor semi-axis defined by (2.14). One expects the mapping $w = f(z)$ to be close to a conformal mapping when $p(z_0)$ is close to unity. The studies started by H. Grötzsch and M. A. Lavrent'ev have shown that to preserve many important properties of analytic functions in the course of quasiconformal mappings we must require that $p(z_0)$ be bounded in G. The class of mappings $\{f(z) = u(x, y) + iv(x, y)\}$ for which the functions $u(x, y)$ and $v(x, y)$ are differentiable in G, the Jacobian $D(u, v)/D(x, y)$ is positive, and

$$p(z) = \frac{A}{2D} + \sqrt{\left(\frac{A}{2D}\right)^2 - 1} \leqslant Q$$

in G (here Q is a finite positive number) is called *Q-quasiconformal* in G.

2.13

HARMONIC AND CONJUGATE HARMONIC FUNCTIONS

In Sec. 6.3 we will show that the real and imaginary parts $u(x, y)$ and $v(x, y)$ of $f(z)$, which is analytic in a certain domain, possess differentiable (hence continuous) partial derivatives of any finite

order. In the present section we will derive several corollaries of this fact. Differentiating the first equation in (2.2) with respect to x and the second with respect to y and noting that $\frac{\partial^2 v}{\partial y\,\partial x}=\frac{\partial^2 v}{\partial x\,\partial y}$, we find, by adding termwise, that

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0.$$

We can obtain a similar equation for $v(x, y)$ if we differentiate the first equation in (2.2) with respect to y and the second with respect to x and subtract termwise the second from the first.

An equation of the type

$$\frac{\partial^2 \varphi}{\partial x^2}+\frac{\partial^2 \varphi}{\partial y^2}=0 \tag{2.16}$$

is a partial differential equation of second order and is known as the *Laplace differential equation.* Functions that in a certain domain have partial derivatives up to the second order inclusive and that satisfy this equation are known as *harmonic functions.* If harmonic functions, in our case u and v, are coupled in this domain by the D'Alembert-Euler equations, they are called *conjugate harmonic.*

Therefore, we can say that *the real and imaginary parts of a function that is analytic in a domain are conjugate harmonic in this domain.* For instance, with respect to $z^2=(x+iy)^2=x^2-y^2+2ixy$, which is analytic in the entire plane, the real part x^2-y^2 and the imaginary part $2xy$ are harmonic in the same plane. At the end of Sec. 2.7 we pointed out that the function $f(z)=\ln|z|+i\operatorname{Arg} z$ is analytic in G: $z\neq 0$; for this reason $\ln|z|$ and $\operatorname{Arg} z$ are harmonic functions in this domain. (This is easily verified by noting that $\ln|z|=\frac{1}{2}\ln(x^2+y^2)$ and $\operatorname{Arg} z=\arctan(y/x)+C$ if $x\neq 0$ or $\operatorname{Arg} z=C-\arctan(x/y)$ if $y\neq 0$.) We note that $\operatorname{Arg} z$ is the simplest example of a many-valued function; strictly speaking, the definition of harmonicity is applicable only to single-valued continuous branches of this function.

Let us suppose that $\varphi(x, y)$ is a (single-valued) function that is harmonic in a given simply connected domain G (a circle, a half-plane, or the entire plane). We will see how to find a function $f(z)=u(x, y)+iv(x, y)$ that is analytic in the domain and whose real part coincides with $\varphi(x, y)$, i.e. $u(x, y)=\varphi(x, y)$. To find the imaginary part we have two equations (the D'Alembert-Euler equations):

$$\frac{\partial v}{\partial x}=-\frac{\partial u}{\partial y}=P(x, y),\quad \frac{\partial v}{\partial y}=\frac{\partial u}{\partial x}=Q(x, y).$$

The functions $P(x, y)$ and $Q(x, y)$ are continuous in G and have continuous first-order partial derivatives (these are expressed in terms of second-order partial derivatives of $u(x, y)$). They also satisfy the

condition

$$\frac{\partial P}{\partial y} = -\frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial x^2} = \frac{\partial Q}{\partial x},$$

in view of which the line integral

$$\int_{(x_0, y_0)}^{(x, y)} P\,dx + Q\,dy$$

does not depend on path from (x_0, y_0) and (x, y) in G and therefore is a single-valued function $\psi(x, y)$ of point (x, y).* This function has the same partial derivatives as the sought function $v(x, y)$:

$$\frac{\partial \psi}{\partial x} = P = \frac{\partial v}{\partial x}, \quad \frac{\partial \psi}{\partial y} = Q = \frac{\partial v}{\partial y},$$

whence $v(x, y)$ may differ from $\psi(x, y)$ only in constant term: $v(x, y) = \psi(x, y) + C$

$$= \int_{(x_0, y_0)}^{(x, y)} P\,dx + Q\,dy + C = \int_{(x_0, y_0)}^{(x, y)} -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy + C$$

(here C is a real number). Using this formula to calculate v, we arrive at two functions differentiable in G:

$$u = \varphi(x, y), \quad v = \psi(x, y) + C,$$

coupled by the D'Alembert-Euler equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

This implies that $f(z) = u(x, y) + iv(x, y) = \varphi(x, y) + i\psi(x, y) + iC$ is analytic in G. Therefore, *knowing a function $\varphi(x, y)$ that is harmonic in a simply connected domain G, we can find an analytic function $f(z)$ whose real part coincides with $\varphi(x, y)$. The sought function is known to within a constant pure imaginary term.*

Example. Suppose G is a domain obtained by deleting the semi-axis $y = 0$, $x \leqslant 0$. It is easy to see that the function $u(x, y) = \frac{1}{2}\ln(x^2 + y^2)$ is harmonic in G. To find the imaginary part we use the equations

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = -\frac{y}{x^2+y^2}, \quad \frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = \frac{x}{x^2+y^2},$$

whence

$$v(x, y) = \int_{(1, 0)}^{(x, y)} -\frac{y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy + C.$$

* See, for example, G. M. Fikhtengol'ts, *Fundamentals of Mathematical Analysis*, vol. 2, Pergamon Press, Oxford, 1965, Sec. 21.4.

For instance, if point $M\ (x,\ y)$ lies in the right half-plane $(x > 0)$, then by integrating along the two-component broken line ANM (Fig. 7) we find that

$$v(x,\ y) = \int_0^y \frac{x\,dy}{x^2+y^2} + C = \arc\tan\frac{y}{x} + C.$$

But if point $M\ (x,\ y)$ lies in the second (or third) quadrant $(x \leqslant 0,\ y \neq 0)$, then by integrating along the broken line $AL'M$ (or $AL''M$, respectively) we find that

$$v(x,\ y) = \int_0^y \frac{dy}{1+y^2} - \int_1^x \frac{y\,dx}{x^2+y^2} + C = \arctan y - \arctan\frac{x}{y} + \arctan\frac{1}{y} + C.$$

Whence we see that $v\ (x,\ y) = \arg z + C$ at all points of G. Therefore

$$f(z) = \frac{1}{2}\ln(x^2+y^2) + i\arg z + iC = \ln|z| + i\arg z + iC.$$

The presented method of finding an analytic function from its real part is applicable in the case of multiply connected domains as well, but in this case the integral

$$\psi\ (x,\ y) = \int_{(x_0,\ y_0)}^{(x,\ y)} P\,dx + Q\,dy$$

at $\partial P/\partial y = \partial Q/\partial x$ is, generally speaking, a many-valued function of point $(x,\ y)$. For this reason the function $f\ (z) = \varphi\ (x,\ y) + i\psi\ (x,\ y) + iC$ is also, generally speaking, many-valued (although $\varphi\ (x, y)$ remains single-valued). This is easily illustrated by finding an analytic function $f\ (z)$ in the domain G: $z \neq 0$ from the given real part $\varphi\ (x,\ y) = \frac{1}{2}\ln\ (x^2 + y^2)$. We arrive at the many-valued function $f\ (z) = \ln\ |\ z\ | + i\ \mathrm{Arg}\ z + iC$.

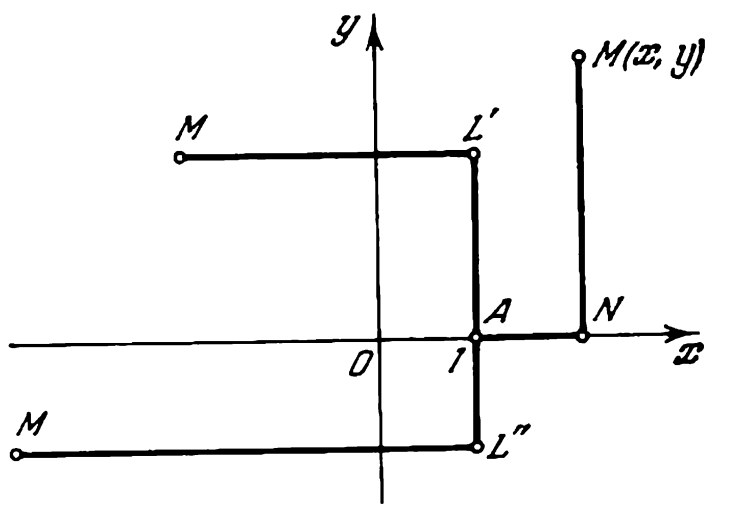

Fig. 7

Finally, we note that we can find analytic functions knowing their imaginary parts in a similar way. Besides, if $\psi\ (x,\ y)$ is the imaginary part of $f\ (z)$, it is the real part of $(1/i)\ f\ (z)$.

Harmonic functions are encountered in many problems of physics and mechanics. For instance, the temperature of a homogeneous plate in thermal equilibrium, the electric potential of a flat conductor,

the velocity potential of a planar laminar flow of a homogeneous incompressible fluid are all harmonic functions of Cartesian coordinates x and y in appropriate domains. The importance of the theory of analytic functions in solving the many problems of physics and mechanics lies in the fact that instead of finding the harmonic functions and working with them we can look for the analytic functions whose real or imaginary parts are these harmonic functions.

2.14
THE HYDROMECHANICAL INTERPRETATION OF AN ANALYTIC FUNCTION

We will study the *steady-state plane-parallel flow of a homogeneous incompressible fluid*. The motion of the fluid is characterized by the fact that the velocity of each point in it is a vector parallel to one and the same plane (x, y) and depends only on the coordinates x, y of the projection of this point on the plane (i.e. does not depend on the third coordinate ζ or on time). In this case it is sufficient to follow the motion of the projections of the fluid particles in plane (x, y), i.e. the motion is *two-dimensional*. In accordance with this we will speak of the motion of the fluid in plane (x, y). Let us suppose that G is the region in the plane occupied by the moving fluid. The closed set F complementary to G with respect to the plane can be considered as a set of projections of cylindrical solids around which the fluid flows in space. We will simply call these projections *solids streamlined by fluids*. In this scheme the solids are at rest, but we can also think of them as moving uniformly and rectilinearly in a motionless fluid. For this it is sufficient, according to the Galilean principle, to impart to the fluid as a whole a constant velocity (in magnitude and direction) equal to that of any point of the particular solid. Then instead of being at rest at infinity the fluid will have the same velocity and the solid can be considered motionless.

Suppose $u(x, y)$ and $v(x, y)$ are the projections of the velocity vector of a particle of the fluid at point (x, y) onto the coordinate axes (these functions are assumed to be continuous). We examine the arc γ of a smooth curve, γ connecting points z_1 and z_2 in G. If ds is the arc element and n the normal to ds directed in such a way that it remains to the right of γ when the arc is traversed from z_1 to z_2, then the area of the parallelogram built on ds and the velocity vector $u + iv$ as sides is ds times the projection of this vector on the normal:

$$[u \cos \widehat{(n, x)} + v \cos \widehat{(n, y)}]\, ds. \tag{2.17}$$

This quantity is positive when the velocity vector and n make an acute angle and negative when they make an obtuse angle. Obviously,

this parallelogram can be considered the base of a right parallelepiped with altitude unity (perpendicular to plane xy). The volume of the parallelepiped coincides in absolute value with the quantity in (2.17), which represents, therefore, the volume of the fluid (with a definite sign) that belongs to a layer of height unity flowing (in unit time) parallel to plane (x, y) through an area that projects itself into ds. The total amount of fluid that belongs to this layer and that flows in unit time through an area which projects itself into γ is

$$\int_{\gamma} [u \cos(\widehat{n, x}) + v \cos(\widehat{n, y})]\, ds.$$

If we note that under these conditions the angle $\widehat{n, x}$ is greater by $3\pi/2$ than the angle $\widehat{t, x}$ between the tangent to γ drawn in the sense of traverse of this curve and the axis, we find that

$$\cos(\widehat{n, x}) = \sin(\widehat{x, t}) = \frac{dy}{ds}.$$

Similarly

$$\cos(\widehat{n, y}) = -\cos(\widehat{x, t}) = -\frac{dx}{ds}.$$

Hence, the above integral can be presented as

$$\int_{\gamma} \left(u \frac{dy}{ds} - v \frac{dx}{ds}\right) ds = \int_{\gamma} -v\, dx + u\, dy. \tag{2.18}$$

This quantity is known as the *flux* of the fluid through arc γ. If the arc is closed and a positive sense is chosen on it in such a way that the interior of the curve is always to the left of the observer when he traverses the arc in this direction, then the normal n is directed to the exterior of γ. For this reason the flux through the element ds of the boundary is positive if the fluid flows out of the interior of γ and negative if it flows into the interior of γ. We assume that the interior of γ belongs to the domain G occupied by the flowing fluid and that there are no *sources* where the fluid could be created and no *sinks* into which the fluid could flow out of the domain. The total flux across γ must therefore be zero:

$$\int_{\gamma} -v\, dx + u\, dy = 0.$$

Applying this reasoning to all closed curves of an arbitrary simply connected subdomain $g \subset G$ without any sources or sinks, we are able to conclude that the fluid flux across any arc g depends not on the shape of the arc but only on the choice of its end points z_1 and z_2.

Suppose that u and v have continuous partial derivatives. Then from the obtained condition it follows that*

$$\frac{\partial u}{\partial x}=\frac{\partial(-v)}{\partial y}. \tag{2.19}$$

We have arrived at the so-called *equation of continuity of an incompressible fluid*. It coincides with one of the D'Alembert-Euler equations for the pair of functions $u(x, y)$ and $-v(x, y)$. In order to arrive at the other D'Alembert-Euler equation, consider the integral

$$\int_{\gamma} u\,dx+v\,dy, \tag{2.20}$$

evaluated along a closed curve γ. The sum $u\,dx + v\,dy$ is the projection of the velocity onto the element ds of γ (strictly speaking, the projection of the velocity onto the tangent in the direction of the sense of traverse of the curve times ds). The integral (2.20) is called the *circulation* around γ. Suppose that in a simply connected subdomain $g \subset G$ the circulation is zero along any closed curve in G. Then, obviously, in this domain the following condition is met:

$$\frac{\partial u}{\partial y}=-\frac{\partial(-v)}{\partial x}. \tag{2.21}$$

This is the second D'Alembert-Euler equation for the pair of functions $u(x, y)$ and $-v(x, y)$. Its physical meaning lies in the fact that it expresses the absence of vorticities in the motion of the fluid (in $g \subset G$). In general, the *vorticity* of vector $u + iv$ in plane motion is a vector perpendicular to the plane (x, y) and having a projection on the third coordinate axis ζ equal to $\partial v/\partial x - \partial u/\partial y$. The vorticity characterizes the rotary motion of fluid particles. If we assume that a fluid particle has hardened, then its angular velocity at point (x, y) is $\frac{1}{2}(\partial v/\partial x - \partial u/\partial y)$. Hence the absence of vorticities in a given domain g implies that at each point in g a fluid particle can have only translational motion and does not experience rotation as a component of motion at the given point. In the absence of vorticities Eq. (2.21) is valid and therefore the vorticity is zero for any closed curve.

If we now assume that in a given subdomain $g \subset G$ Eq. (2.19) (the absence of sources and sinks) and Eq. (2.21) (the absence of vorticities in the fluid) are valid, we find that the function

$$u + (-v)\,i = u - iv,$$

which is the conjugate complex of the velocity of a fluid particle, is an analytic function of point $z = x + iy$.

* See the footnote on p. 50.

In Sec. 5.7 we will show that each function that is analytic in a simply connected domain in the finite plane can be considered the derivative of a single-valued analytic function.

Since g is a simply connected domain, $u - iv$ can be considered as the derivative of a single-valued function $f(z)$ analytic in g and defined to within an arbitrary constant term. This function, which satisfies the condition

$$f'(z) = u - iv,$$

is known as the *complex potential* or the *characteristic function of a flow*.

We set

$$f(z) = \varphi(x, y) + i\psi(x, y). \tag{2.22}$$

Then

$$f'(z) = \frac{\partial\varphi}{\partial x} + i\frac{\partial\psi}{\partial x} = \frac{\partial\psi}{\partial y} - i\frac{\partial\varphi}{\partial y}$$

and hence

$$\frac{\partial\varphi}{\partial x} = u, \quad \frac{\partial\varphi}{\partial y} = v, \quad \frac{\partial\psi}{\partial x} = -v, \quad \frac{\partial\psi}{\partial y} = u.$$

The first pair of these relations shows that $\varphi(x, y)$, the real part of the complex potential, is the *velocity potential* of the motion. Obviously we can write it as

$$\varphi(x, y) - \varphi(x_0, y_0) = \int_{(x_0, y_0)}^{(x, y)} u\,dx + v\,dy.$$

From the second pair it follows that

$$\psi(x, y) - \psi(x_0, y_0) = \int_{(x_0, y_0)}^{(x, y)} -v\,dx + u\,dy,$$

i.e. the difference between two values of $\psi(x, y)$ is the flux of the fluid across any curve that links the points for which the difference is taken. The function $\psi(x, y)$, the imaginary part of the complex potential, is the *stream function*.

Let us consider two families of curves:

$$\varphi(x, y) = \text{const}, \tag{2.23}$$

$$\psi(x, y) = \text{const}. \tag{2.24}$$

In the plane of values of the function $\zeta = f(z)$ these families are depicted by coordinate axes $\xi = \text{const}$ and $\eta = \text{const}$. Since these are orthogonal, the families (2.23) and (2.24) are orthogonal as well due to the fact that the mapping $\zeta = f(z)$ is conformal (the statement holds only where $f'(z) \neq 0$ holds, i.e. where the particle's velocity is nonzero).

The curves in (2.23) have the peculiar feature that

$$\frac{\partial \varphi}{\partial x} dx + \frac{\partial \varphi}{\partial y} dy = 0,$$

i.e.

$$u\, dx + v\, dy = 0. \tag{2.25}$$

These are *equipotential curves*. On the other hand, curves (2.24) have the property that

$$\frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy = 0,$$

i.e.

$$-v\, dx + u\, dy = 0$$

or

$$\frac{dx}{u} = \frac{dy}{v}. \tag{2.26}$$

These are called *streamlines*.

Equation (2.25) implies that the velocity vector $u + iv$ at points where it is nonzero is directed along the normal to the corresponding equipotential curve. In the same way, it follows from Eq. (2.26) that this vector is directed along the tangent to the appropriate streamline. Hence we have again proved that equipotential curves and streamlines are mutually perpendicular.

Moreover, from the fact that the velocity vector coincides with the tangent to a streamline and that the motion is steady-state (velocities depend only on the position of a fluid particle) it follows that streamlines coincide with particle trajectories.*

If the domain G in which the fluid flows has isolated point sources, sinks, or vorticities (i.e. points corresponding to nonzero vorticities), then by excluding these from G we arrive at a multiply connected domain G', where in each simply connected subdomain the above reasoning remains valid. This implies, as before, that the function $u - iv$ conjugate complex to velocity $u + iv$ is single-valued in G' and analytic.

The function (2.22) (in general many-valued) is again the complex potential of the fluid motion, but in each simply connected subdomain $g' \subset G'$ it splits into single-valued analytic branches. Since the derivative of each coincides with the same function $u - iv$, the different branches of the complex potential may differ only by a constant term.

Therefore we have established that each plane-parallel steady-state flow in a domain G of an incompressible fluid has corresponding to it a certain function $f(z)$, the complex potential, that is analytic

* In the general case of unsteady flow we can speak of streamlines as curves that at every moment of time coincide with the velocity vectors; however, the streamlines in this case do not, in general, coincide with the particle trajectories.

at all points of G except at those where there are sources, sinks, or vorticities. This function is in general many-valued, but its derivative, which at each point of the domain is conjugate complex to the velocity $u + iv$, is single-valued.

The domain's boundary can be considered as the collection of outlines (projections or cross sections) of the walls of the vessel with the fluid or the outlines of the cylindrical bodies streamlined by the fluid. Since the fluid particles in contact with the vessel walls must still flow, the boundary of the domain must be included in the system of streamlines.

More generally, if in a certain domain G we have an analytic (except at some points) function $f(z)$ in general many-valued but with a single-valued derivative $f'(z)$, then this function can be interpreted as the complex potential of a fluid flow in G. The points in G at which the analyticity of $f(z)$ breaks down can be interpreted as sources, sinks, or vorticities in the flow, and the domain boundary as the outlines of the bodies streamlined by the flow. For the last interpretation to be possible it is necessary for the boundary of G to be included in the system of streamlines, i.e. the stream function ($\psi(x, y) = \operatorname{Im} f(z)$) must have a constant value at all boundary curves (at each curve a different value).

2.15

EXAMPLES

We illustrate the general ideas of the previous section by two examples. Let us consider first an entire linear function

$$f(z) = az.$$

This function can be considered a complex potential of fluid flow over the entire plane. The flow is translational and its velocity is $\overline{f'(z)} = \bar{a}$ at every point. Assuming that $a = \alpha + i\beta$, for the velocity potential we can write

$$\varphi(x, y) = \alpha x - \beta y,$$

and for the stream function

$$\psi(x, y) = \beta x + \alpha y.$$

Figure 8 shows the streamlines and the equipotential curves orthogonal to the streamlines.

If instead of the entire plane we take a strip bounded by two straight lines parallel to $\bar{a}$, the same function represents the complex potential of fluid flow along the strip.

The other example is

$$f(z) = z^2.$$

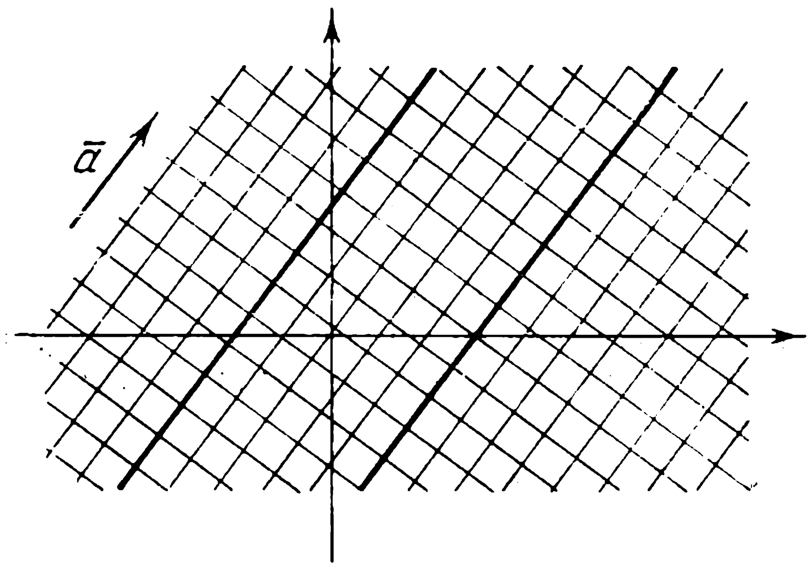

Fig. 8

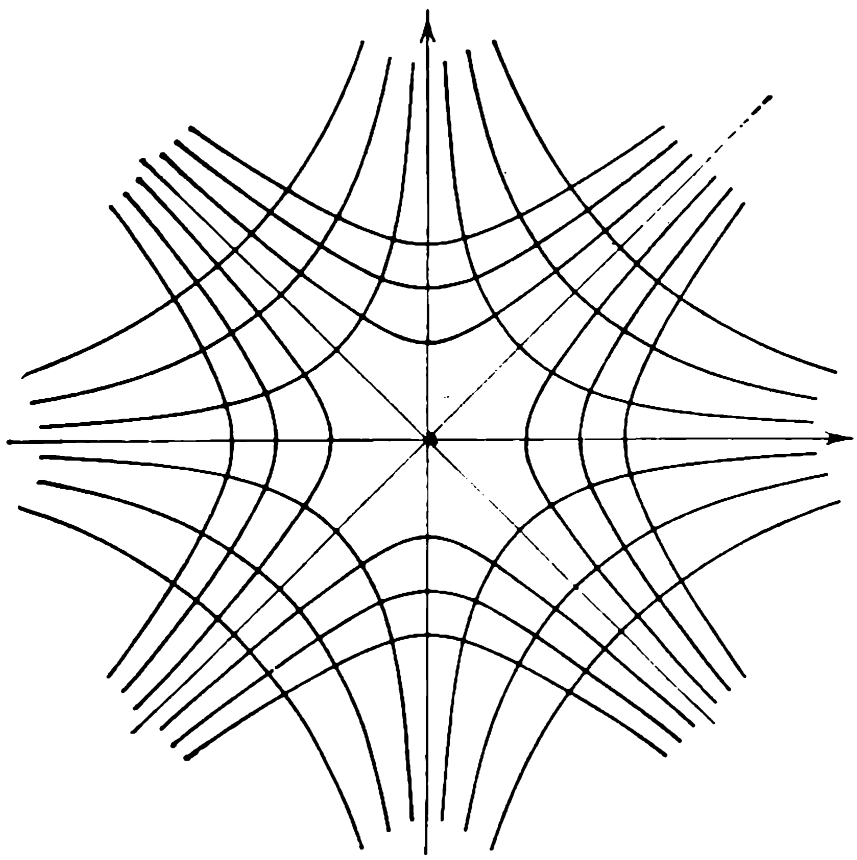

Fig. 9

This is a complex potential for the flow of a fluid occupying the entire plane. The velocity of a fluid particle at point z is $\overline{f'(z)} = 2\bar{z}$, the velocity potential is

$$\varphi(x, y) = x^2 - y^2,$$

and the stream function is

$$\psi(x, y) = 2xy.$$

Figure 9 shows the equipotential curves $x^2 - y^2 = \text{const}$ and the streamlines $2xy = \text{const}$ for the given flow. These are obviously

equilateral hyperbolas. The two coordinate axes also belong to the streamlines ($2xy = 0$). Where the axes intersect (the origin of coordinates) the velocity is zero. If instead of the entire plane we take one quadrant (e.g. the first), the same function represents the complex potential of plane-parallel fluid flow in the first quadrant. The sides of the angle then represent the vessel walls between which the fluid flows.

3
ELEMENTARY ANALYTIC FUNCTIONS AND THE CORRESPONDING CONFORMAL MAPPINGS

3.1
THE POLYNOMIAL

The simplest and the most important class of differentiable functions is the functions that are single-valued and analytic in the entire plane without the point at infinity. Such functions are called *entire*. A good example of an entire function is the polynomial,

$$a_0 + a_1 z + a_2 z^2 + \ldots + a_n z^n = P_n(z).$$

If $n = 0$, this is a constant. But if $n > 0$ and $a_n \neq 0$, then $\lim_{z \to \infty} P_n(z) = \infty$. Hence, the polynomial of a degree higher than zero turns into ∞ at the point at infinity.

If w is an arbitrary (proper) complex number, then, as is known from algebra, the equation $P_n(z) = w$ has n roots, some of which may be the same (multiple roots). For this reason each point in the w-plane belongs to the image of the z-plane through the mapping $w = f(z)$, and each point in the w-plane has n preimages $z_1, z_2, \ldots, z_n$. In addition, $P_n(\infty) = \infty$ and, hence, ∞ belongs to the image of the extended plane. The preimages of the point at infinity $w = \infty$ are the roots of the polynomial equation $P_n(z) = \infty$, i.e. also the point at infinity. For the sake of symmetry we will consider this point an n-tuple root. Hence, *a polynomial of degree n ($a_n \neq 0$, $n > 0$) maps the extended complex plane into itself in a way such that to each image point w there corresponds n preimages $z_1, z_2, \ldots, z_n$.* However, as we have said before, for some exceptional values of w ($w = \infty$ included) the number of preimages may be less than n.

It is easily seen that the number of these exceptional points cannot be greater than n. Indeed, if $w_0 \neq \infty$ and the polynomial equation $P_n(z) = w_0$ has multiple roots, then for each of them $P_n'(z) = 0$ (a fact known from algebra). But the latter equation has $n - 1$ roots (among which there may be multiple ones) $\zeta_1, \zeta_2, \ldots, \zeta_{n-1}$. This suggests that w_0 must have one of the following $n - 1$ values:

$P_n(\zeta_1), \ldots, P_n(\zeta_{n-1})$. If to this we add the point at infinity, we will obtain the (greatest possible) n points in the w-plane that have less than n preimages in the z-plane.

3.2
THE POINTS AT WHICH CONFORMAL MAPPING DOES NOT WORK

In view of the general theory the mapping $w = P_n(z)$ is conformal at all points except those points $\zeta_1, \zeta_2, \ldots, \zeta_{n-1}$ at which the derivative vanishes and, perhaps, point $z = \infty$.

In the case where $n = 1$ the polynomial is an entire linear function and the mapping is one-to-one and conformal in the extended plane, i.e. the point at infinity included (see Secs. 2.10 and 2.11). At $n > 1$ the conformality breaks down at the above-mentioned points.

Indeed, let $P'_n(z_0) = 0$. Then $z = z_0$ is a multiple root of the equation $P_n(z) - P_n(z_0) = 0$; therefore we can write

$$P_n(z) - P_n(z_0) = (z - z_0)^k Q(z),$$

where $k \geqslant 2$ is the multiplicity of the root $z = z_0$ (the number k, as is known, is a unity greater than the multiplicity of the root $z = z_0$ of the equation $P'_n(z) = 0$), and the polynomial $Q(z)$ does not vanish at $z = z_0$. Assuming that $P_n(z) = w$ and $P_n(z_0) = w_0$, we find that

$$\operatorname{Arg}(w - w_0) = \operatorname{Arg}(z - z_0)^k + \operatorname{Arg} Q(z),$$

whence

$$\lim_{z \to z_0} [\operatorname{Arg}(w - w_0) - \operatorname{Arg}(z - z_0)^k] = \operatorname{Arg} Q(z_0).$$

Now let us suppose that $z = \lambda(t)$ is a curve L starting at point z_0 (where $z_0 = \lambda(t_0)$) and having at this point a tangent that is inclined to the real axis by an angle

$$\operatorname{Arg} \lambda'(t_0) = \lim_{t_1 \to t_0,\ t_1 > t_0} \operatorname{Arg} \frac{z_1 - z_0}{t_1 - t_0}$$

(see Sec. 2.8). The image curve Λ in the w-plane is $w = P_n[\lambda(t)] = \mu(t)$. This does not directly imply that there exists a tangent to Λ at point w_0 $(t = t_0)$ (since $\mu'(t_0) = P'_n(z_0)\lambda'(t_0) = 0$). But the slope of the secant passing through points w_0 and $w \neq w_0$ is (for $t > t_0$)

$$\operatorname{Arg} \frac{w - w_0}{t - t_0} = \operatorname{Arg} \frac{w - w_0}{(z - z_0)^k} + k \operatorname{Arg} \frac{z - z_0}{t - t_0}$$
$$\to \operatorname{Arg} Q(z_0) + k \operatorname{Arg} \lambda'(t_0) \text{ at } t \to t_0,$$

which shows that a tangent does exist.

If L_1 and L_2 are two curves $z = \lambda_1(t)$ and $z = \lambda_2(t)$ starting at point z_0 and forming at this point an angle

$$\theta = \operatorname{Arg} \lambda_2'(t_2) - \operatorname{Arg} \lambda_1'(t_1) \quad (\lambda_1(t_1) = \lambda_2(t_2) = z_0),$$

then the image curves Λ_1 and Λ_2 start at point w_0 and form at this point an angle

$$[\operatorname{Arg} Q(z_0) + k \operatorname{Arg} \lambda_2'(t_2)] - [\operatorname{Arg} Q(z_0) + k \operatorname{Arg} \lambda_1'(t_1)]$$
$$= k[\operatorname{Arg} \lambda_2'(t_2) - \operatorname{Arg} \lambda_1'(t_1)] = k\theta.$$

Therefore, *the function $w = P_n(z)$ maps all angles with vertices at points where the derivative $P_n'(z)$ vanishes into angles that are greater by the multiplicity of the corresponding root of equation $P_n(z) - P_n(z_0) = 0$.*

Applying the mapping $\zeta = 1/z$, the reader can easily verify that at $n > 1$ the conformality breaks down also at the point at infinity. Namely, all angles with their vertices at the point at infinity increase n-fold under the mapping $w = P_n(z)$.

3.3 THE MAPPING $w = (z - a)^n$

Let us consider the particular case of the mapping $w = (z - a)^n$ $(n > 1)$. The function maps the extended plane into itself in a way such that each point w has n preimages in the z-plane. Exceptions are the points $w = 0$ and $w = \infty$, at which the preimages merge into one point, a and ∞ respectively. The preimages z $(\neq a, \neq \infty)$ can be determined from the equation

$$w = (z - a)^n,$$

so that

$$z = a + \sqrt[n]{w} = a + \sqrt[n]{|w|}_{+} \cos\left(\frac{\operatorname{Arg} w}{n} + i \sin \frac{\operatorname{Arg} w}{n}\right).$$

Obviously, these n points are arranged at the vertices of a regular n-gon with its center at a.

The mapping $w = (z - a)^n$ is conformal at all points except $z = a$ and $z = \infty$. The angles with vertices at these two points increase n-fold under this mapping.

To have a clearer idea about this mapping, we note that

$$|w| = |z - a|^n \text{ and } \operatorname{Arg} w = n \operatorname{Arg}(z - a).$$

This implies that each circle of radius r with the center at $z = a$ is mapped into a circle of radius r^n with the center at point $w = 0$. If point z traverses the circle $|z - a| = r$ one time in the positive sense (i.e. $\operatorname{Arg}(z - a)$ continuously grows, and increases by 2π after completion of the circle), point w traverses the circle $|w| = r^n$ in

the same direction n times (i.e. Arg w continuously grows and increases by $2n\pi$). Let us now make point z traverse the ray Arg $(z-a) = \varphi_0 + 2k\pi$ from a to ∞. Our formulas show that the corresponding point w will in the process traverse the ray Arg $w = n\varphi_0 + 2m\pi$ from the origin of coordinates to infinity.

Consider the domain g that is the interior of the angle θ, where $0 < \theta \leqslant 2\pi/n$, with its vertex at point a. Suppose this angle is restricted by rays

$$\operatorname{Arg}(z-a) = \varphi_0 + 2k\pi,$$
$$\operatorname{Arg}(z-a) = \varphi_1 + 2m\pi$$
$$(\varphi_1 - \varphi_0 = \theta).$$

From the aforementioned we can conclude that the image of g in the w-plane is a domain d that is the interior of an angle $n\theta$ with its

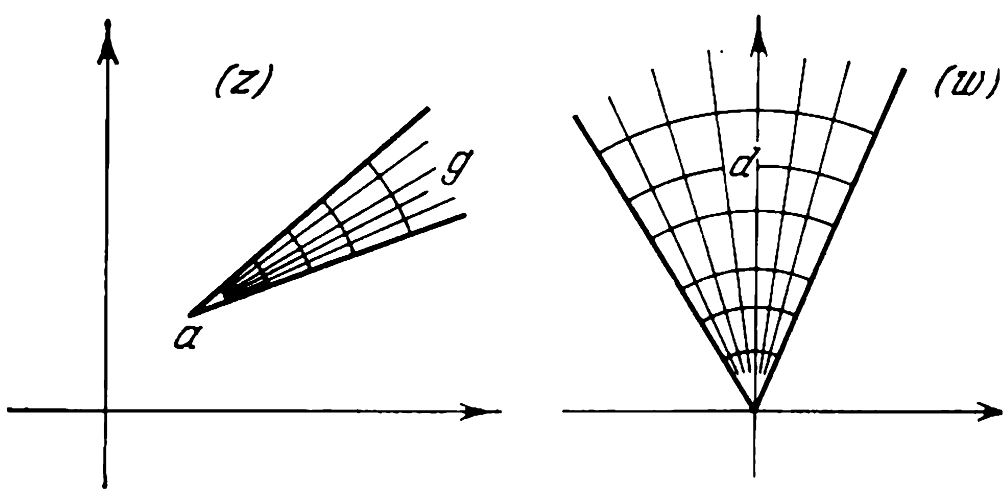

Fig. 10

vertex at the origin of coordinates and restricted by rays (Fig. 10). The correspondence between g and d established by means of the function $w = (z-a)^n$ is one-to-one. Indeed, since the function $w = (z-a)^n$ is single-valued, to verify the above statement it is sufficient to establish that each point $w \in d$ has only one preimage in g. To this end we note that all the n preimages of point w are situated in the z-plane at the vertices of a regular n-gon with its center at a, so that two of them can occur inside an angle with its vertex at a only if the angle is greater than $2\pi/n$. But the angle represented by g does not exceed $2\pi/n$, whereby g has only one preimage of each point in d. This completes the proof of the proposition.

Therefore, *the function $w = (z-a)^n$ maps in a one-to-one manner and conformally the interior of any angle with rectilinear sides, the vertex, at a, and an opening span θ $(0 < \theta \leqslant 2\pi/n)$ into the interior of an angle with rectilinear sides, the vertex at the origin of coordinates, and an opening span $n\theta$.*

For this reason whenever we must map an angle with rectilinear sides into another angle severalfold greater, we resort to the function $w = (z - a)^n$.

There is no reason to think, however, that the mapping $w = (z-a)^n$ $(n > 1)$ carries any straight line into a straight line and any circle into a circle. Assume for a moment that $a = 0$ and $n = 2$. Then the function is $w = z^2$. What is the result of mapping, by means of the function $w=z^2$, of straight lines that do not pass through the origin of coordinates and are parallel to one of the coordinate axes. Let us take a line that is parallel to the imaginary axis: $z=c+it$, $c \neq 0$, $-\infty < t < +\infty$. The image curve is $w = (c + it)^2$ or assuming that $w = u + iv$ and separating the real and imaginary parts, we have

$$u = c^2 - t^2, \ v = 2ct$$

$$(-\infty < t < \infty).$$

These are the equations of the image curve in Cartesian coordinates in parametric form. Solving for t, we obtain

$$v^2 = 4c^2 (c^2 - u).$$

This is the equation of a parabola with its axis directed along the real axis in the negative direction, its focus at the origin of coordinates, and parameter $p = 2c^2$. In a similar manner we see that a straight line parallel to the real axis, $z = t + ic'$, is mapped into the parabola

$$v^2 = 4c'^2 (u + c'^2)$$

with its axis directed along the real axis in the positive direction, its focus at the origin of coordinates, and parameter $p' = 2c'^2$. Therefore, *two families of straight lines parallel to the coordinate axes are mapped by means of the function $w = z^2$ into two families of parabolas with a common focus at the origin of coordinates and with axes directed along the real axis* (Fig. 11). From the fact that the families of straight lines are mutually orthogonal and the mapping is conformal it follows that the obtained families of parabolas are also mutually orthogonal; this can be verified by direct computations, too.

The reader must bear in mind, however, that the mapping of the entire z-plane by means of the function $w - z^2$ is not one-to-one because each point w that is not zero or ∞ has two preimages. Specifically, the preimages of the parabola $v^2 = 4c^2 (c^2 - u)$ are two straight lines symmetric with respect to the imaginary axis: $z = c + it$ and $z = - c + it$. In the same way the preimages of the parabola $v^2 = 4c'^2 (u + c'^2)$ are two straight lines symmetric with respect to the real axis: $z = t + ic'$ and $z = t - ic'$. But if we

consider only the image of a half-plane g bounded by a straight line passing through the origin of coordinates (such a half-plane is the interior of an angle with its vertex at the origin of coordinates and

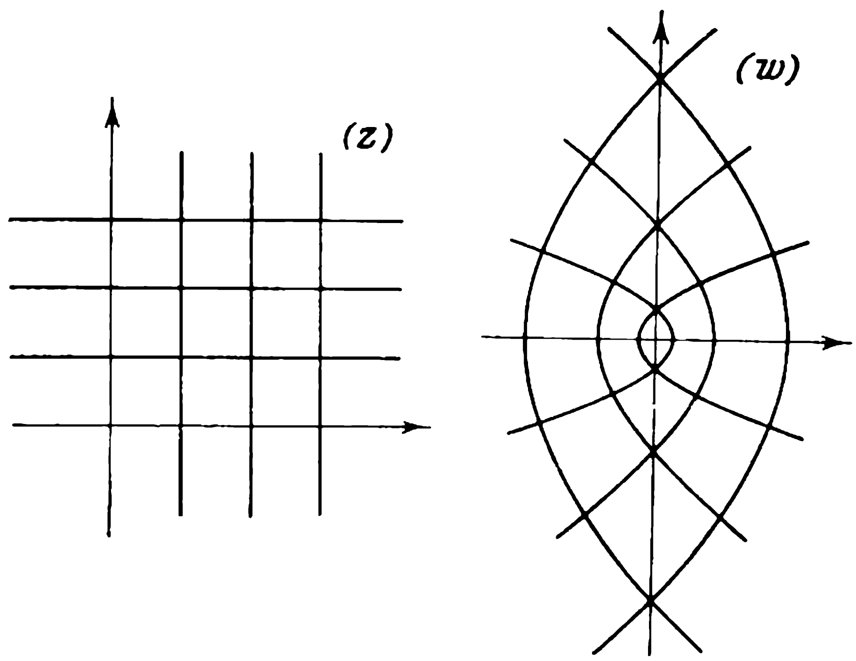

Fig. 11

an opening span of π), then in agreement with the aforesaid the correspondence between g and its image d is one-to-one: d then represents an angle of 2π with the vertex at the origin of coordinates, and with two sides that merge into a ray starting at the origin.

3.4

THE GROUP PROPERTIES OF LINEAR-FRACTIONAL TRANSFORMATIONS

In Sec. 2.10 we established that the linear-fractional function $w = L(z) = (az + b)/(cz + d)$ with $ad - bc \neq 0$ carries out a one-to-one and conformal mapping of the extended complex plane into itself. Here we will study the properties of this mapping, which is called *linear-fractional*, and is the best-known and most widely used type of conformal mapping.

We start by defining the set M of all linear-fractional mappings (or transformations) with nonzero determinants. Two mappings,

$$L_1(z) = \frac{a_1 z + b_1}{c_1 z + d_1} \quad \text{and} \quad L_2(z) = \frac{a_2 z + b_2}{c_2 z + d_2},$$

will be considered identical if and only if $L_1(z) = L_2(z)$ for all values of z. For this to be so it is sufficient that the corresponding coefficients be proportional to each other, namely

$$a_2 = \lambda a_1, \; b_2 = \lambda b_1, \; c_2 = \lambda c_1, \; d_2 = \lambda d_1 \quad (\lambda \neq 0),$$

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which are also the necessary conditions. Indeed, if $L_1(z) = L_2(z)$, then

$$L_1(0) = L_2(0),\ L_1(1) = L_2(1),\ \text{and}\ L_1(\infty) = L_2(\infty).$$

This means that

$$\frac{b_1}{d_1} = \frac{b_2}{d_2} = p,\quad \frac{a_1+b_1}{c_1+d_1} = \frac{a_2+b_2}{c_2+d_2},\quad \frac{a_1}{c_1} = \frac{a_2}{c_2} = q.$$

Substituting

$$b_1 = d_1 p,\ b_2 = d_2 p,\ a_1 = c_1 q,\ \text{and}\ a_2 = c_2 q$$

into the second equation, we obtain

$$\frac{c_1 q + d_1 p}{c_1+d_1} = \frac{c_2 q + d_2 p}{c_2+d_2},\ \text{or}\ (c_1 d_2 - c_2 d_1)(q-p) = 0.$$

But $q \neq p$ (otherwise we would have $a_1/c_1 = b_1/d_1$, or $a_1 d_1 - b_1 c_1 = 0$, which is contradictory to the assumption). Hence,

$$\frac{c_1}{d_1} = \frac{c_2}{d_2}.$$

The obtained relationships can be rewritten as

$$\frac{a_2}{a_1} = \frac{b_2}{b_1} = \frac{c_2}{c_1} = \frac{d_2}{d_1} = \lambda,$$

which is what we set out to prove.

From what has been said it follows that the value of the determinant of a linear-fractional mapping is not in itself a characteristic of such a mapping. Indeed, as we go over from the coefficients a_1, b_1, c_1, and d_1 to the coefficients λa_1, λb_1, λc_1, and λd_1 ($\lambda \neq 0$), the determinant is multiplied by λ^2. But at least this determinant remains nonzero if it was different from zero at the beginning.

The mapping

$$U(z) = z,$$

which obviously belongs to the set M, will be called the *identity mapping* (or *identity function*).

A mapping that is the *inverse* (*inverse mapping*, or *inverse function*) of

$$z_1 = L(z) = \frac{az+b}{cz+d}$$

is such that for each value of z_1 the image is the preimage of z in the given mapping. The inverse is

$$z = \frac{dz_1 - b}{-cz_1 + a}$$

and is denoted by L^{-1}.

If

$$z_1 = L(z) = \frac{az+b}{cz+d} \text{ and } z_2 = L_1(z_1) = \frac{a_1z_1+b_1}{c_1z_1+d_1}$$

are two arbitrary linear mappings (with nonzero determinants, as usual), the mapping obtained by performing them successively in a definite order is called the *product* of these mappings. Let us first perform the mapping $z_1 = L(z)$ and then the mapping $z_2 = L_1(z_1)$. Then their product is denoted by $L_1L(z)$:

$$z_2 = L_1L(z) = \left[a_1\frac{az+b}{cz+d} + b_1\right] \Big/ \left[c_1\frac{az+b}{cz+d} + d_1\right]$$
$$= [(aa_1+cb_1)z + (ba_1+db_1)]/[(ac_1+cd_1)z + (bc_1+dd_1)].$$

Hence, $z_2 = L_1L(z)$ is also a linear-fractional mapping and its determinant is

$$(aa_1+cb_1)(bc_1+dd_1) - (ba_1+db_1)(ac_1+cd_1)$$
$$= (ad-bc)(a_1d_1-b_1c_1) \neq 0.$$

We see that $L_1L(z)$ belongs to M.

Obviously,

$$LL^{-1}(z) = L^{-1}L(z) = U(z).$$

Note that the mapping $z_2 = LL_1(z)$, which is the result of first applying $z_1 = L_1(z)$ and then $z_2 = L(z_1)$, differs, in general, from $L_1L(z)$. For instance, if

$$L(z) = \frac{z}{z+1} \text{ and } L_1(z) = \frac{z+1}{z-1},$$

then

$$L_1L(z) = -2z-1 \text{ and } LL_1(z) = \frac{z+1}{2z}.$$

The defined operation of multiplication is associative, i.e. for any linear mappings L, L_1, and L_2 we have

$$(LL_1)L_2 = L(L_1L_2).$$

This property can easily be verified. Indeed, let $L_2(z) = z_2$. Then

$$(LL_1)L_2(z) = LL_1[L_2(z)] = LL_1(z_2)$$

and

$$L(L_1L_2)(z) = L[L_1L_2(z)] = LL_1(z_2);$$

whence

$$(LL_1)L_2(z) = L(L_1L_2)(z).$$

The associativity property is extended to multiplication of any number of mappings and saves us the bother of keeping track of

brackets in products. For instance,

$$L\,[L_1\,(L_2L_3)]\,(z) = LL_1\,(L_2L_3)\,(z) = L\,(L_1L_2)\,L_3\,(z) = \dots = LL_1L_2L_3\,(z).$$

Since for any two elements (mappings) of set M there exists their product L_1L (or LL_1), also an element of M, and for every element L of M there exists an inverse L^{-1}, also in M, we call M a *mapping group* (a *transformation group*)*.

3.5
THE CIRCULAR PROPERTY

Let us prove the so-called *circular property* of linear-fractional mappings, which reflects the fact that the image of a straight line or circle under the mapping $w = L\,(z)$ is either a straight line or a circle, and the image of a straight line is either a straight line or a circle, just as the image of a circle may be a straight line or a circle.

For the entire linear function $L\,(z) = \alpha z + \beta$ this property is quite obvious, since the mapping $w = L\,(z)$ is reduced to a translation (at $\alpha = 1$) or a rotation and expansion (at $\alpha \neq 1$); see Sec. 2.10.

Now let us examine the mapping

$$w = \Lambda\,(z) = \frac{1}{z},$$

which we have used many times. Obviously the equation of a straight line or a circle can be written in general form as

$$A\,(x^2 + y^2) + 2Bx + 2Cy + D = 0.$$

We have a straight line at $A = 0$ and for B and C not zero simultaneously, and a circle at $A \neq 0$ and $B^2 + C^2 - AD > 0$. Substituting $z\bar{z}$ for $x^2 + y^2$, $z + \bar{z}$ for $2x$, and $-i\,(z - \bar{z})$ for $2y$, we write this equation as

$$Az\bar{z} + (B - iC)z + (B + iC)\bar{z} + D = 0,$$

or

$$Az\bar{z} + \bar{E}z + E\bar{z} + D = 0, \tag{3.1}$$

where $E = B + iC$. Here $A = 0$ and the complex number E is nonzero in the case of a straight line, and $A \neq 0$ and $E\bar{E} - AD > 0$ in the case of a circle. Conversely, any equation of such a type with real coefficients A and D and complex coefficients E and $\bar{E}$ is an equation of a straight line if $A = 0$ and $E \neq 0$ and an equation of a circle if $A \neq 0$ and $E\bar{E} - AD > 0$. To verify this fact it is suffi-

* See, for example, A. G. Kurosh, *Higher Algebra*, Mir Publishers, Moscow, 1972 (reprinted in 1975 and 1980), Sec. 63.

cient to go over from z to x and y:

$$z\bar{z} = x^2 + y^2, \ z = x + iy, \ \bar{z} = x - iy.$$

If we wish to find the image curve for (3.1) under the mapping $w = 1/z$, we must substitute $1/w$ for z in Eq. (3.1). This yields

$$A\frac{1}{w\bar{w}} + \bar{E}\frac{1}{w} + E\frac{1}{\bar{w}} + D = 0, \text{ or } Dw\bar{w} + Ew + \bar{E}\bar{w} + A = 0. \qquad (3.2)$$

Equation (3.2) has the same form as (3.1), with D substituted for A, A for D, and $\bar{E}$ for E. This implies that at $D = 0$ this is the equation of a straight line (since either $A = 0$ and $E \neq 0$ or $A \neq 0$ and $E\bar{E} - AD = E\bar{E} > 0$, i.e. again $E \neq 0$) and at $D \neq 0$ this is the equation of a circle (since at $A \neq 0$ Eq. (3.1) represents a circle and, hence, the condition $E\bar{E} - AD > 0$ is met, and at $A = 0$ Eq. (3.1) represents a straight line and, hence, $E \neq 0$, which implies that $E\bar{E} - AD = E\bar{E} > 0$). We have proved that the *image of a straight line or a circle under the mapping $w = \Lambda(z) = 1/z$ is a straight line or a circle.*

Turning to the case of an arbitrary linear-fractional function

$$w = L(z) = \frac{az+b}{cz+d} \quad (c \neq 0),$$

we write it in the form

$$w = \frac{a}{c} + \frac{bc-ad}{c(cz+d)}.$$

Suppose that

$$z_1 = L_1(z) = cz + d, \ z_2 = \Lambda(z_1) = \frac{1}{z_1}, \quad \text{and} \quad w = L_2(z_2) = \frac{a}{c} + \frac{bc-ad}{c} z_2.$$

Then $L(z)$ can be written as a product of three mappings:

$$L = L_2 \Lambda L_1.$$

Since the image of a straight line or a circle under each of the three mappings L_1, Λ, and L_2 is a straight line or a circle, the mapping L possesses the same property. We have thus proved the circular property of a linear-fractional mapping.

If $c \neq 0$, then at point $\delta = -d/c$ the function $L(z) = (az+b)/(cz+d)$ turns into ∞. For this reason the image of a straight line or circle passing through δ must contain the point at infinity $L(\delta) = \infty$ and, hence, cannot be a circle. Thanks to the circular property the image curve is a straight line. The image curve not passing through point δ cannot contain the point at infinity and, hence, cannot be a straight line. Thanks to the circular property the image curve is

a circle. Therefore, *all straight lines and circles passing through point* δ *are mapped under the function* $w = L(z)$ *into straight lines in the* w*-plane, and the straight lines and circles not passing through* δ *are mapped into circles in this plane.*

Let $w = L(z)$ be an arbitrary linear-fractional function, γ a straight line or a circle in the z-plane, and Γ the image curve in the w-plane (i.e. also a straight line or a circle). Consider the domains g_1 and g_2 bounded by γ in the z-plane; these are either two half-planes or the exterior and interior of a circle. We will see that the image of one domain is a domain in the w-plane bounded by Γ, while the image of the other is the other domain in the w-plane bounded by Γ. Indeed, let z_1 be a point in g_1 and z_2 a point belonging to g_2. Since z_1 and

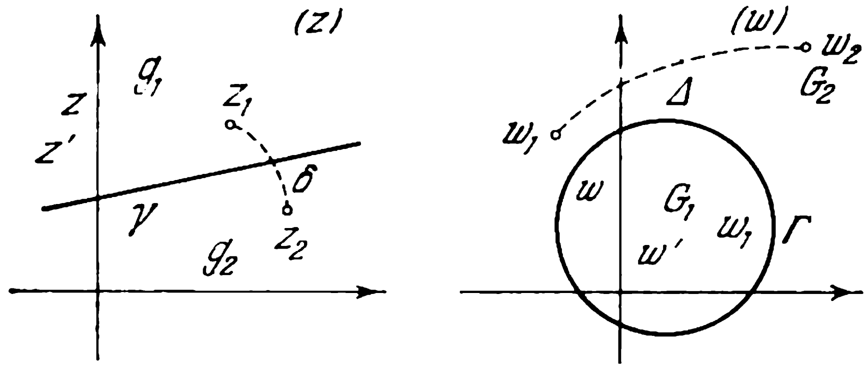

Fig. 12

z_2 do not lie on γ, the images of these points cannot lie on Γ and, hence, fall into the domains into which Γ separates the w-plane. If we assume for a moment that the images fall into one domain, they can be linked by a straight line (or an arc of a circle) Δ without any common points with Γ (Fig. 12). The preimage of Δ in the z-plane must be a straight line (or an arc of a circle) δ connecting z_1 and z_2 and having no common points with γ. But this is impossible since z_1 and z_2 lie in different domains g_1 and g_2. Therefore, from the fact that z_1 and z_2 belong to different domains g_1 and g_2 it follows that their images w_1 and w_2 also belong to different domains bounded by Γ. We denote the domain containing w_1 by G_1 and that containing w_2 by G_2. Let us now show that G_1 is the image of g_1 and G_2 the image of g_2.

Indeed, if z' is a point in g_1, then from the fact that z' and z_2 belong to different domains g_1 and g_2 it follows that, as proved, their images w' and w_2 belong to different domains G_1 and G_2. But w_2 belongs to G_2, therefore w' belongs to G_1. Thus, the image of each point in g_1 belongs to G_1 and in the same way the image of each point in g_2 belongs to G_2. Finally, we take an arbitrary point w in G_1. It must be an image of one of the points, z, in g_1 or g_2. But z cannot belong to g_2 since otherwise w would be a point in G_2. Therefore, w is the image of a point z belonging to g_1. Thus G_1 is the image of g_1 and G_2 the

image of g_2. We have proved that two domains bounded by a curve γ are mapped into two domains bounded by curve Γ and have established that to find which of the two domains bounded by Γ is the image of a given domain g_1 bounded by γ it is sufficient to follow the image w_1 of one point $z_1 \in g_1$. The domain G_1 to which this image belongs is the image of g_1.

3.6
THE INVARIANCE OF THE ANHARMONIC RATIO

If a linear-fractional mapping $w = L(z)$ is not an identity mapping, then w differs from z in general. But even in this case there exist so-called *fixed points of a mapping*. These are defined as points at which

$$z = L(z) = \frac{az+b}{cz+d}.$$

Let us first put $c = 0$ $(d \neq 0)$. Then $L(z)$ is the entire linear function

$$l(z) = \alpha z + \beta \quad \left(\alpha = \frac{a}{d},\ \beta = \frac{b}{d}\right).$$

Since $l(\infty) = \infty$, one of the fixed points of the entire linear mapping is the point at infinity $z = \infty$. At $\alpha \neq 1$ there is another fixed point determined from the equation

$$z = \alpha z + \beta.$$

This is the point $z = \beta/(1-\alpha)$. At $\alpha = 1$ and $\beta \neq 0$ there is no finite fixed point. But if $\alpha \neq 1$, $\beta \neq 0$, and $\alpha \to 1$, then the finite fixed point $\beta/(1-\alpha)$ tends to the point at infinity. For this reason, in the case of the mapping $l(z) = z + \beta$ $(\beta \neq 0)$ the point at infinity can be thought of as two fixed points that have merged.

Now we put $c \neq 0$. Then $L(\infty) = a/c \neq \infty$, i.e. the point $z = \infty$ is not fixed. The point $-d/c$ is also not fixed, since

$$L\left(-\frac{d}{c}\right) = \infty \neq -\frac{d}{c}.$$

Assuming that $z \neq \infty$ and $z \neq -d/c$, we solve the equation

$$z = \frac{az+b}{cz+d},$$

or

$$cz^2 - (a-d)z - b = 0.$$

This yields

$$z = \frac{a-d \pm \sqrt{(a-d)^2+4bc}}{2c}.$$

If $(a-d)^2 + 4bc \neq 0$, we are able to find two different (finite) fixed points. But if $(a-d)^2 + 4bc = 0$, these two points merge into one (finite) fixed point $(a-d)/2c$.

Hence, *a linear-fractional mapping that differs from the identity mapping has only two fixed points, which in a particular case can merge into one.* Entire linear transformations are fully characterized by the fact that at least one fixed point is the point at infinity.

A linear-fractional mapping that has more than two fixed points can only be the identity mapping $U(z) = z$ (all points of which are in essence fixed). This implies that for two linear-fractional mappings $L(z)$ and $\Lambda(z)$ to coincide it is sufficient for the equation $L(z) = \Lambda(z)$ be valid for three different points, z_1, z_2, and z_3. Indeed, let $L(z_k) = \Lambda(z_k) = w_k$ $(k = 1, 2, 3)$. Then $\Lambda^{-1}(w_k) = z_k$ $(k = 1, 2, 3)$ and, hence, the mapping $\Lambda^{-1} L(z)$ carries the points z_k into the same points: $\Lambda^{-1}L(z_k) = z_k$ $(k = 1, 2, 3)$ (since $L(z_k) = w_k$ and $\Lambda^{-1}(w_k) = z_k$). We have established that the mapping $\Lambda^{-1}L$ has three different fixed points, z_1, z_2, and z_3, and therefore is an identity mapping:

$$\Lambda^{-1}L = U.$$

Consequently,

$$\Lambda(\Lambda^{-1}L) = \Lambda U.$$

But

$$\Lambda(\Lambda^{-1}L) = (\Lambda\Lambda^{-1})L = UL = L$$

and

$$\Lambda U = \Lambda.$$

We finally obtain

$$L = \Lambda,$$

which is what we set out to prove.

Therefore, *to define a linear-fractional transformation it is sufficient to know three points, w_1, w_2, w_3, into which three given points z_1, z_2, z_3 are mapped.*

But how does one find the linear-fractional transformation? We first assume that z_1, z_2, and z_3 are three finite points and that $w_1 = 0$, $w_2 = \infty$, and $w_3 = 1$.

For the linear-fractional function

$$w = \Lambda(z) = \frac{az+b}{cz+d}$$

to vanish at $z = z_1$ and become infinite at $z = z_2$, it is necessary and sufficient that $z = z_1$ be the zero of the numerator $az + b$, i.e. $az + b = a(z - z_1)$, and $z = z_2$ be the zero of the denominator, i.e. $cz + d = c(z - z_2)$. The sought function must therefore be of the form

$$w = \frac{a}{c}\,\frac{z-z_1}{z-z_2}.$$

But w must be equal to unity at $z = z_3$. From the equation

$$1 = \frac{a}{c}\,\frac{z_3-z_1}{z_3-z_2}$$

we obtain

$$\frac{a}{c}=1\Big/\frac{z_3-z_1}{z_3-z_2},$$

whence

$$w=\Lambda(z)=\frac{z-z_1}{z-z_2}\Big/\frac{z_3-z_1}{z_3-z_2}.$$

This is the sought for linear-fractional function that carries points z_1, z_2, z_3 into the points 0, ∞, and 1, respectively.

Now suppose that w_1, w_2, and w_3 are three arbitrary (but finite) and different points and $w = L(z)$ is the linear-fractional mapping that satisfies the criteria

$$L(z_1) = w_1, \quad L(z_2) = w_2, \text{ and } L(z_3) = w_3.$$

As we have just proved, the function $\zeta = \Lambda_1(w) = \frac{w-w_1}{w-w_2}\Big/\frac{w_3-w_1}{w_3-w_2}$ maps the points w_1, w_2, and w_3 into the points 0, ∞, and 1. For this reason the function $\Lambda_1 L(z)$ maps the points z_1, z_2, and z_3 into the points 0, ∞, and 1, i.e.

$$\Lambda_1 L(z)=\Lambda(z)=\frac{z-z_1}{z-z_2}\Big/\frac{z_3-z_1}{z_3-z_2}.$$

From

$$\Lambda_1 L = \Lambda$$

it follows that

$$\Lambda_1^{-1}(\Lambda_1 L) = \Lambda_1^{-1}\Lambda,$$

i.e.

$$L = \Lambda_1^{-1}\Lambda$$

(since $\Lambda_1^{-1}(\Lambda_1 L) = (\Lambda_1^{-1}\Lambda_1)L = UL = L$).

The last relationship solves the problem, since the mappings Λ and Λ_1 are known:

$$\Lambda(\zeta)=\frac{\zeta-z_1}{\zeta-z_2}\Big/\frac{z_3-z_1}{z_3-z_2}, \quad \Lambda_1(\zeta)=\frac{\zeta-w_1}{\zeta-w_2}\Big/\frac{w_3-w_1}{w_3-w_2}.$$

But in studying the mapping $w = L(z)$ it is more convenient to use directly the relationship

$$\Lambda_1 L(z) = \Lambda(z),$$

from which after substituting w for $L(z)$ we obtain

$$\Lambda_1(w) = \Lambda(z),$$

or

$$\frac{w-w_1}{w-w_2}\Big/\frac{w_3-w_1}{w_3-w_2}=\frac{z-z_1}{z-z_2}\Big/\frac{z_3-z_1}{z_3-z_2}. \tag{3.3}$$

This equation gives the linear-fractional function $w = L(z)$ in implicit form.

We have solved our problem under the assumption that all points z_1, z_2, z_3, w_1, w_2, and w_3 are finite points. For instance, if $z_1 = \infty$, the function $\Lambda(z)$, which maps points $z_1 = \infty$, z_2, and z_3 into points $w_1 = 0$, $w_2 = \infty$, and $w_3 = 1$, assumes the form*

$$\Lambda(z) = \frac{1}{z - z_2} \Big/ \frac{1}{z_3 - z_2}.$$

For this reason Eq. (3.3) is substituted by the equation

$$\frac{w - w_1}{w - w_2} \Big/ \frac{w_3 - w_1}{w_3 - w_2} = \frac{1}{z - z_2} \Big/ \frac{1}{z_3 - z_2} \tag{3.4}$$

(under the assumption that w_1, w_2, and w_3 are finite). If $z_2 = \infty$, the function $\Lambda(z)$, which maps points z_1, $z_2 = \infty$, and z_3 into point $w_1 = 0$, $w_2 = \infty$, and $w_3 = 1$, assumes the form

$$\Lambda(z) = (z - z_1)/(z_3 - z_1).$$

For this reason Eq. (3.3) is substituted by the equation

$$\frac{w - w_1}{w - w_2} \Big/ \frac{w_3 - w_1}{w_3 - w_2} = (z - z_1)/(z_3 - z_1). \tag{3.5}$$

If $z_3 = \infty$, the function $\Lambda(z)$, which maps points z_1, z_2, and $z_3 = \infty$ into points $w_1 = 0$, $w_2 = \infty$, and $w_3 = 1$, assumes the form

$$\Lambda(z) = \frac{z - z_1}{z - z_2}$$

and Eq. (3.3) is substituted by

$$\frac{w - w_1}{w - w_2} \Big/ \frac{w_3 - w_1}{w_3 - w_2} = \frac{z - z_1}{z - z_2}. \tag{3.6}$$

In the same manner the left-hand side of Eq. (3.3) must be substituted by

$$\frac{1}{w - w_2} \Big/ \frac{1}{w_3 - w_2}, \quad (w - w_1)/(w_3 - w_1), \quad \frac{w - w_1}{w - w_2}$$

depending on whether $w_1 = \infty$ or $w_2 = \infty$ or $w_3 = \infty$.

As a result we arrive at the following mnemonic rule: if $z_k = \infty$ or $w_l = \infty$ (k, $l = 1, 2, 3$), then in Eq. (3.3) the differences that contain z_k or w_l must be changed to unity. The reader can easily

* We arrive at this formula if, assuming z_1 to be finite, we rewrite

$$\frac{z - z_1}{z - z_2} \Big/ \frac{z_3 - z_1}{z_3 - z_2}$$

in the form

$$\frac{\dfrac{z}{z_1} - 1}{z - z_2} \Bigg/ \frac{\dfrac{z_3}{z_1} - 1}{z_3 - z_2}$$

and then go over to the limit as $z_1 \to \infty$.

verify the validity of this rule by applying the passage to the limit in Eq. (3.3) as $z_k \to \infty$ or $w_l \to \infty$.

An important general property of linear-fractional mappings follows from Eq. (3.3). Let a, b, c, and d be arbitrary (finite) and different complex numbers. The ratio

$$\frac{c-a}{c-b} \Big/ \frac{d-a}{d-b}$$

is called the *anharmonic ratio* (*cross ratio*, or *double ratio*) of these four numbers or points a, b, c, and d and is denoted by $[a, b; c, d]$:

$$[a, b; c, d] = \frac{c-a}{c-b} \Big/ \frac{d-a}{d-b}.$$

We extend the definition of an anharmonic ratio to the case where one of the four points a, b, c, or d is at infinity. Namely, by the anharmonic ratio of four points in which one point is at infinity we imply such a limit of the anharmonic ratio of four finite points in which three coincide with the given finite points and the fourth tends to the point at infinity. According to this definition,

$$[\infty, b; c, d] = \frac{1}{c-b} \Big/ \frac{1}{d-b},$$

$$[a, \infty; c, d] = (c-a)/(d-a),$$

$$[a, b; \infty, d] = 1 \Big/ \frac{d-a}{d-b},$$

$$[a, b; c, \infty] = \frac{c-a}{c-b}.$$

Now suppose that $w = L(z)$ is an arbitrary linear-fractional mapping. By A, B, C, and D we denote the points into which the four finite points are mapped under this function. Since three points a, b, and d are mapped into points A, B, and D, the function $L(z)$ and z are linked by Eq. (3.3):

$$\frac{w-A}{w-B} \Big/ \frac{D-A}{D-B} = \frac{z-a}{z-b} \Big/ \frac{d-a}{d-b},$$

where we must substitute unity for those differences where the point at infinity is present. Assuming that $z = c$, we must put $w = C$ (since in our mapping point C corresponds to point c). Hence

$$\frac{C-A}{C-B} \Big/ \frac{D-A}{D-B} = \frac{c-a}{c-b} \Big/ \frac{d-a}{d-b}$$

(differences in which the point at infinity is present must be changed to unity), or

$$[A, B; C, D] = [a, b; c, d]. \tag{3.7}$$

Therefore, *under a linear-fractional transformation, the anharmonic ratio of four points does not change*; *in other words, the anharmonic ratio is the invariant of a linear mapping.*

3.7

MAPPING DOMAINS BOUNDED BY STRAIGHT LINES OR CIRCLES

Relying on the circular property of linear-fractional mappings and the possibility of mapping any triple of points z_1, z_2, and z_3 into a given triple of points w_1, w_2, w_3, we can prove the following proposition:

For any straight lines or circles γ and Γ and two triples of points z_1, z_2, z_3 and w_1, w_2, w_3 that belong to γ and Γ, respectively, there exists a linear-fractional function $w = L(z)$ that maps γ into Γ in a way such that z_1, z_2, z_3 map into w_1, w_2, w_3, respectively.

Indeed, let us build the linear-fractional function $w = L(z)$ that satisfies the condition $L(z_j) = w_j$ $(j = 1, 2, 3)$. We already know that such a function exists and is the only function that satisfies these conditions. It maps a straight line or a circle γ into a straight line or a circle Γ'. But γ passes through the points z_1, z_2, and z_3 and, hence, Γ' passes through the points w_1, w_2, and w_3. Since through three points one cannot draw more than one circle or straight line Γ' coincides with Γ. Therefore, $w = L(z)$ satisfies all the conditions of the proposition.

Let us again take arbitrary straight lines or circles γ and Γ (which may differ or be the same) and suppose g is one of the two domains bounded by γ and G one of the two domains bounded by Γ. Obviously, g and G may be a half-plane, the interior of a circle, or the exterior of a circle. We choose an arbitrary triple of points z_1, z_2, and z_3 on γ and for definiteness assume that as an observer moves along γ from point z_1 to point z_2 to point z_3, the domain g is to the left. Now let w_1, w_2, and w_3 be a triple of points on Γ so that when an observer moves along Γ from w_1 to w_2 to w_3, the domain G remains to the left. In all other respects the points w_1, w_2, and w_3 are arbitrary. We build, as we did before, a linear-fractional function $w = L(z)$ that satisfies the condition $w_j = L(z_j)$ $(j = 1, 2, 3)$ and, hence, maps γ into Γ. We wish to show that this function also maps g into G. Indeed, if δ is a segment of a normal to γ passing through point z_2 into domain g, i.e. to the left of an observer at point z_2 looking along γ in the above-defined direction, then in view of the conformality of the mapping $w = L(z)$ the image Δ of this segment (either a segment of a straight line or an arc) is also to the left of an observer at point w_2 looking along Γ in the above-defined direction (Fig. 13). Therefore Δ belongs to G. Thus we have established that

domain G contains images of some points that belong to g (namely, the images of points on δ). But according to Sec. 3.5, $L(g)$ is one of the domains whose boundary coincides with the image of the boundary of g, i.e. with $\Gamma = L(\gamma)$. Since there are only two such domains and one of them, G, contains the images of points in g, this domain is the sought for image of g, i.e. $G = L(g)$.

Here is an example. We wish to map conformally the upper half-plane $y > 0$ into the interior of a unit circle. We choose $z_1 = -1$,

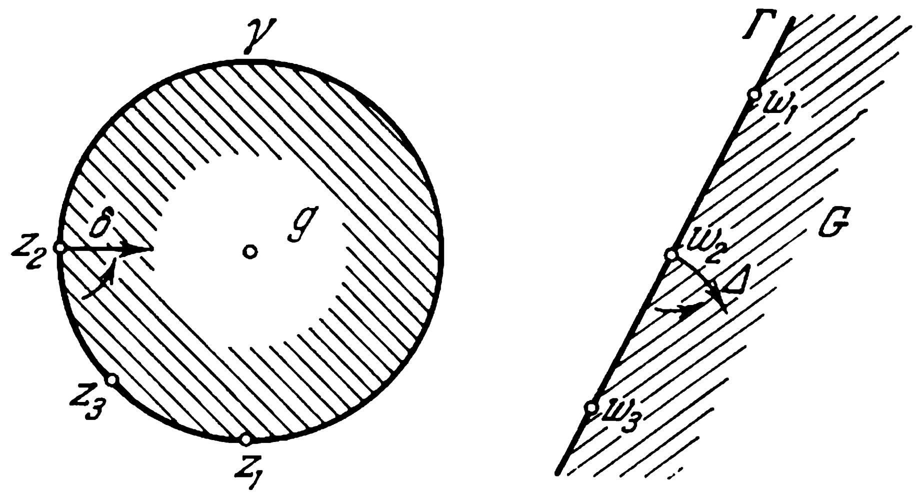

Fig. 13

$z_2 = 0$, and $z_3 = 1$, so that the half-circle is to the left of an observer going along the real axis from z_1 to z_2 to z_3, and then choose three points on the unit circle, w_1, w_2, w_3, so that the interior of the circle remains to the left of an observer going along the circle from w_1 to w_2 to w_3. For simplicity we assume that $w_1 = 1$, $w_2 = i$, and $w_3 = -1$. Then the linear mapping satisfying the conditions $w_j = L(z_j)$ $(j = 1, 2, 3)$ is the one we are looking for. It can be represented in the form

$$\frac{w-1}{w-i}\Big/\frac{-1-1}{-1-i}=\frac{z+1}{z}\Big/\frac{1+1}{1}$$

or

$$w=\frac{z-i}{iz-1}.$$

3.8

SYMMETRY AND ITS CONSERVATION

Suppose z_1 and z_2 are two points symmetric with respect to a straight line γ. Then the center of an arbitrary circle δ passing through z_1 and z_2 lies on γ and, hence, δ is orthogonal to γ. The straight line passing through z_1 and z_2 is also orthogonal to γ. It is easily seen that the converse statement is also true: if any circle or straight

line passing through a pair of points z_1 and z_2 is orthogonal to a straight line γ, then z_1 and z_2 are symmetric with respect to γ.

Let us map the complex z-plane by means of a linear-fractional function $w = L(z)$ in a way such that a straight line γ is carried into a straight line or circle Γ. Then a pair of points z_1 and z_2 symmetric with respect to γ is mapped into a pair of points w_1 and w_2, and each circle or straight line δ passing through z_1 and z_2 is mapped into a circle or straight line Δ passing through w_1 and w_2. Conversely, any straight line or circle passing through w_1 and w_2 is the image of a straight line or circle passing through z_1 and z_2. By virtue of the symmetry of points z_1 and z_2 with respect to γ and the conformality of the mapping $w = L(z)$, all straight lines and circles passing through w_1 and w_2 are orthogonal to Γ. If, hence, Γ is a straight line, points $w_1 = L(z_1)$ and $w_2 = L(z)_2$ are symmetric with respect to Γ. Generalizing the idea of symmetry, we will call two points *symmetric with respect to a circle* if each straight line or circle passing through these points is orthogonal to the given circle. We can then say that if a straight line γ is mapped by means of a linear-fractional function $w = L(z)$ into a circle Γ, each pair of points symmetric with respect to the straight line is mapped into a pair of points symmetric with respect to the circle, and vice versa. This suggests that knowing the circle Γ and the point w_1, we can uniquely determine the point w_2 symmetric to w_1 with respect to Γ. Indeed, if we assume for a moment that there exist two different points w_2 and w_2' symmetric to w_1 with respect to Γ, then under the mapping by means of a linear-fractional function of Γ into γ the points w_1, w_2, and w_2' would be carried into the points z_1, z_2, and $z_2' \neq z_2$, where z_1 and z_2 and also z_1 and z_2' are symmetric with respect to γ, which is obviously impossible.

The above reasoning concerning the mapping of a straight line γ into a circle or straight line Γ is carried over without change to the case where a circle is mapped into a circle. As a result, we can say that under a linear-fractional mapping each pair of points symmetric with respect to circle γ is carried into a pair of points symmetric with respect to circle Γ, the image of γ. We have thus arrived at the following general property of *symmetry conservation* under linear-fractional mappings.

If points z_1 and z_2 are symmetric with respect to a straight line or circle γ, then for every linear-fractional mapping $w = L(z)$ their images w_1 and w_2 are symmetric with respect to the image of γ, i.e. $\Gamma = L(\gamma)$.

Note a particular case of this statement. Suppose that γ is mapped into circle Γ and that z_1 is a point carried under this mapping into the center w_1 of Γ. Then point z_2 symmetric to z_1 is mapped into point w_2 of the extended plane, a point symmetric to w_1 with respect to Γ. But this is the point at infinity. Indeed, a straight line connecting w_1 and $w_2 = \infty$, i.e. any line passing through the center of the circle Γ, is orthogonal to Γ.

Since a symmetric point is unique (for a given point and with respect to a given circle), ∞ is the only point symmetric to the center of circle Γ with respect to Γ.

Suppose that γ is a straight line or circle. The mapping of the extended plane that results in each point z being carried into a point z^* symmetric to z with respect to γ is called a *symmetry mapping* with respect to γ, or *mirror reflection* with respect to γ. In the case where γ is a circle we also speak of *inversion* with respect to γ.

Let us give an analytic expression for a symmetry mapping. First we assume that γ is a straight line. It is fully defined by a point a on it and a unit vector $\alpha = \cos\theta + i\sin\theta$ directed along it.

We perform the linear transformation

$$z = a + \alpha w = l(w),$$

which obviously maps the real axis in the w-plane into γ. Since the mapping $w = l^{-1}(z)$ carries γ into the real axis, each pair of points z and z^* symmetric with respect to γ is mapped as a result into a pair of points w and w^* symmetric with respect to the real axis. This pair of image points is therefore conjugate complex numbers: $w = t$ and $w^* = \bar{t}$. Consequently, $z - a = \alpha t$, or $\overline{z - a} = \overline{\alpha t} = \alpha^{-1}\bar{t}$, and $z^* - a = \alpha w^* = \alpha\bar{t}$. Solving these two equations for $\bar{t}$, we obtain

$$z^* - a = \alpha^2\,\overline{(z - a)}. \tag{3.8}$$

This equation shows that to perform a symmetry mapping with respect to a straight line γ passing through point a at an angle θ to the real axis, we must go over from vector $z - a$ to vector $\overline{z - a}$, which is symmetric to the former with respect to the real axis, and then rotate the latter about point a through an angle 2θ.

Now let us consider the symmetry mapping with respect to a circle Γ with center at a and a radius R $(0 < R < \infty)$. We perform a linear-fractional mapping that carries Γ into the real axis. The simplest mapping of this type is

$$z = a + R\,\frac{1+iw}{1-iw} = L(w),$$

which relates the points $w_1 = -1$, $w_2 = 0$, and $w_3 = 1$ on the real axis to points on the circle Γ, i.e. $z_1 = a - iR$, $z_2 = a + R$, and $z_3 = a + iR$, and, hence, maps the real axis into Γ. The inverse mapping $w = L^{-1}(z)$ carries Γ into the real axis and each pair of points z and z^* symmetric with respect to Γ into the pair w and w^* symmetric with respect to the real axis. Since w and w^* are depicted by conjugate complex numbers $w = t$ and $w^* = \bar{t}$, we see that $z - a = R\,\dfrac{1+it}{1-it}$, or $\overline{z - a} = R\,\dfrac{1-i\bar{t}}{1+i\bar{t}}$, and $z^* - a = R\,\dfrac{1+i\bar{t}}{1-i\bar{t}}$.

Multiplying termwise the last two relationships, we find that

$$\overline{(z-a)}\,(z^*-a)=R^2$$

or

$$z^*-a=\frac{R^2}{\overline{z-a}}. \tag{3.9}$$

In particular, the symmetry mapping with respect to the unit circle $|z|=1$ has the form $z^*=1/\bar{z}$. (This mapping was considered in Sec. 1.4.)

From (3.9) it follows, first, that

$$\operatorname{Arg}(z^*-a)=-\operatorname{Arg}\overline{(z-a)}=\operatorname{Arg}(z-a)$$

and, second, that

$$|z^*-a|\,|z-a|=R^2.$$

Consequently, the points z and z^* lie on a ray emerging from the center of Γ at distances from the center such that their product is equal to R^2. These two conditions, or (which is the same) Eq. (3.9), fully define the position of one of the points z or z^* by that of the other point, i.e. inversion with respect to the circle $|z-a|=R$. The reader is advised to verify the above-proved property of points symmetric with respect to a circle by elementary geometric reasoning.

From Eq. (3.8) or Eq. (3.9) it follows that *the general symmetry mapping is reduced to successive multiplication of a linear (entire or fractional) mapping and a symmetry mapping with respect to the real axis.*

For instance, symmetry mapping with respect to a straight line can be written as

$$z_1=\bar{a}+\bar{\alpha}^2(z-a) \tag{3.10}$$

and

$$z^*=\bar{z}_1,$$

and symmetry mapping with respect to a circle as

$$z_1=\bar{a}+\frac{R^2}{z-a} \tag{3.11}$$

and

$$z^*=\bar{z}_1.$$

Since linear mapping is conformal and possesses the circular property and symmetry mapping with respect to the real axis has the same properties, the only difference being that, while preserving the magnitudes of angles, the latter mapping changes their direction to the opposite, the symmetry mapping in the general case possesses the same properties (of conformality and the circular property).

Namely, *the general symmetry mapping is a conformal mapping of the second kind and carries straight lines and circles into straight lines and circles.*

3.9

EXAMPLES

Two examples illustrate the use of symmetry conservation under linear-fractional mapping.

Example 1. We wish to map the upper half-plane conformally into the interior of the circle $|w| < R$ in a way such that a given point α of the half-plane is carried into the circle center $w = 0$.

The sought for function, if it exists, vanishes at $z = \alpha$, i.e. $L(\alpha) = 0$. Hence we know the zero $z = \alpha$ of function $L(z)$. But point $\bar{\alpha}$ symmetric to α with respect to the real axis must be mapped by this function into a point that is symmetric to the center of the circle with respect to the circle, i.e. the point at infinity. For this reason $L(z)$ has the form

$$w = L(z) = \frac{a(z-\alpha)}{c(z-\bar{\alpha})} = \lambda \frac{z-\alpha}{z-\bar{\alpha}}, \tag{3.12}$$

where λ is a nonzero complex number.

Let us show that the obtained function maps the half-plane into the circle $|w| < |\lambda|$ in a way such that point α is carried into the center of the circle $w = 0$. The last condition, obviously, is satisfied by function (3.12) for any λ. We need only check whether the real axis is mapped into the circle of radius $|\lambda|$ with its center at the origin of coordinates. True, since if $z = x$ is a real number, the numbers $x - \alpha$ and $x - \bar{\alpha}$ are conjugate complex and, hence,

$$|w| = |L(x)| = \left|\lambda \frac{x-\alpha}{x-\bar{\alpha}}\right| = |\lambda| \left|\frac{x-\alpha}{x-\bar{\alpha}}\right| = |\lambda|.$$

We have established that the images of all points on the real axis lie on the circle $|w| = |\lambda|$, whence it follows, due to the circular property, that the image of the real axis is a circle.

To obtain a mapping of the half-plane into a circle of radius R we must put $|\lambda| = R$. The argument of λ, however, remains indeterminate. The geometric meaning of this fact is quite clear. A transition in (3.12) from one value of λ to another while $|\lambda|$ remains equal to R is equivalent to a change in the arguments of all points by the same quantity, i.e. a rotation of the circle about its center $w = 0$. Under such rotation the circle is carried into itself, its center remains in place, and the conditions of the problem are not altered.

If we want the problem to have a unique solution, we must introduce an additional requirement. For instance, we can require that

(a) a given point on the real axis, $x = x_0$, be mapped into point $w = R$ on the circle, or

(b) the derivative $L'(\alpha)$ be a real, positive number (geometrically this means that the tangents to curves passing through point α must not change their slope under the mapping).

Indeed, under condition (a) from (3.12) we obtain

$$R = L(x_0) = \lambda \frac{x_0 - \alpha}{x_0 - \bar{\alpha}},$$

whence

$$\lambda = R \frac{x_0 - \bar{\alpha}}{x_0 - \alpha}$$

and

$$L(z) = R \frac{x_0 - \bar{\alpha}}{x_0 - \alpha} \frac{z - \alpha}{z - \bar{\alpha}}. \tag{3.13}$$

Obviously,

$$|\lambda| = R \left| \frac{x_0 - \bar{\alpha}}{x_0 - \alpha} \right| = R.$$

Now let $\alpha = \xi + i\eta$, where $\eta > 0$. Since

$$L'(\alpha) = \frac{\lambda}{\alpha - \bar{\alpha}},$$

under condition (b) we conclude that $\lambda/(\alpha - \bar{\alpha}) = \lambda/2\eta i$, which means that λ/i is a real, positive number. But, on the other hand, $|\lambda|$ must be equal to R. For this reason $\lambda = iR$ and

$$L(z) = iR \frac{z - \alpha}{z - \bar{\alpha}}. \tag{3.14}$$

Example 2. We wish to map the circle $|z| < R$ conformally into itself in a way such that a given point $z = \alpha$ on this circle is mapped into its center $w = 0$.

The sought for linear-fractional function $L(z)$, if it exists, vanishes at $z = \alpha$, i.e. $L(\alpha) = 0$. Hence we know the zero $z = \alpha$ of function $L(z)$. But point α^* symmetric to α with respect to the circle $|z| = R$ must be mapped by this function into a point that is symmetric to the center with respect to the same circle, i.e. the point at infinity. The linear-fractional function, therefore, has the form

$$w = L(z) = \lambda \frac{z - \alpha}{z - \alpha^*},$$

where λ is a nonzero complex number. According to (3.9), the point α^* symmetric to point α with respect to the circle $|z| = R$ is

$$\alpha^* = \frac{R^2}{\bar{\alpha}}.$$

Therefore

$$w = L(z) = -\lambda\bar{\alpha}\,\frac{z-\alpha}{R^2-\bar{\alpha}z} = \mu\,\frac{z-\alpha}{R^2-\bar{\alpha}z}. \tag{3.15}$$

Let us show that the obtained function maps the circle $|z| < R$ into the circle $|w| < |\mu|/R$ in a way such that point α is carried into the center of the circle $w = 0$. The last condition, obviously, is satisfied by function (3.15) for any μ. We need only check whether the circle $|z| = R$ is mapped by means of (3.15) into the circle $|w| = |\mu|/R$. True, since if we assume that $z = R\zeta$, where $\zeta = \cos\theta + i\sin\theta$ $(0 \leqslant \theta < 2\pi)$, is a point on the circle $|z| = R$, its image

$$w = L(R\zeta) = \mu\,\frac{R\zeta-\alpha}{R^2-\bar{\alpha}R\zeta} = \frac{\mu}{R\zeta}\,\frac{R\zeta-\alpha}{R\bar{\zeta}-\bar{\alpha}}$$

has the modulus

$$|w| = \left|\frac{\mu}{R\zeta}\right|\left|\frac{R\zeta-\alpha}{R\bar{\zeta}-\bar{\alpha}}\right| = \frac{|\mu|}{R}$$

(since $|\zeta| = 1$ and $|(R\zeta-\alpha)/(R\bar{\zeta}-\bar{\alpha})| = 1$). This means that the circle $|w| = |\mu|/R$ is the image of circle $|z| = R$.

To obtain a mapping of the circle of radius R into itself, we must obviously put $|\mu| = R^2$ in (3.15). The argument of μ remains indeterminate. For the problem to have a unique solution, we must impose one of the following additional requirements:

(a) a given point a on the circle $|z| = R$ is mapped into a point $w = R$ on the same circle;

(b) the derivative $L'(\alpha)$ is a real, positive number.

It is left to the reader to verify that in case (a)

$$L(z) = R\,\frac{R^2-\bar{\alpha}a}{a-\alpha}\,\frac{z-\alpha}{R^2-\bar{\alpha}z}, \tag{3.16}$$

while in case (b)

$$L(z) = \frac{R^2(z-\alpha)}{R^2-\bar{\alpha}z}. \tag{3.17}$$

3.10

THE ZHUKOVSKII (JOUKOWSKI) FUNCTION

Consider the function

$$w = \frac{1}{2}\left(z+\frac{1}{z}\right) = \lambda(z).$$

This function was named the *Zhukovskii* (or *Joukowski*) *function* after N. E. Zhukovskii, a Russian applied mathematician and aerodynamicist who was the first to use it widely in aerodynamics.

Obviously, for any z the equation $w = \frac{1}{2}(z + 1/z)$ has no more than two roots, z_1 and z_2. Since $\lambda(z) = \lambda(1/z) = w$, we find that $z_1 z_2 = 1$, i.e. if one root belongs to the interior of the unit circle, the other belongs to the exterior, and vice versa. Consequently, the sets of values $w = \lambda(z)$ assumed in the interior and exterior of the unit circle must be the same.

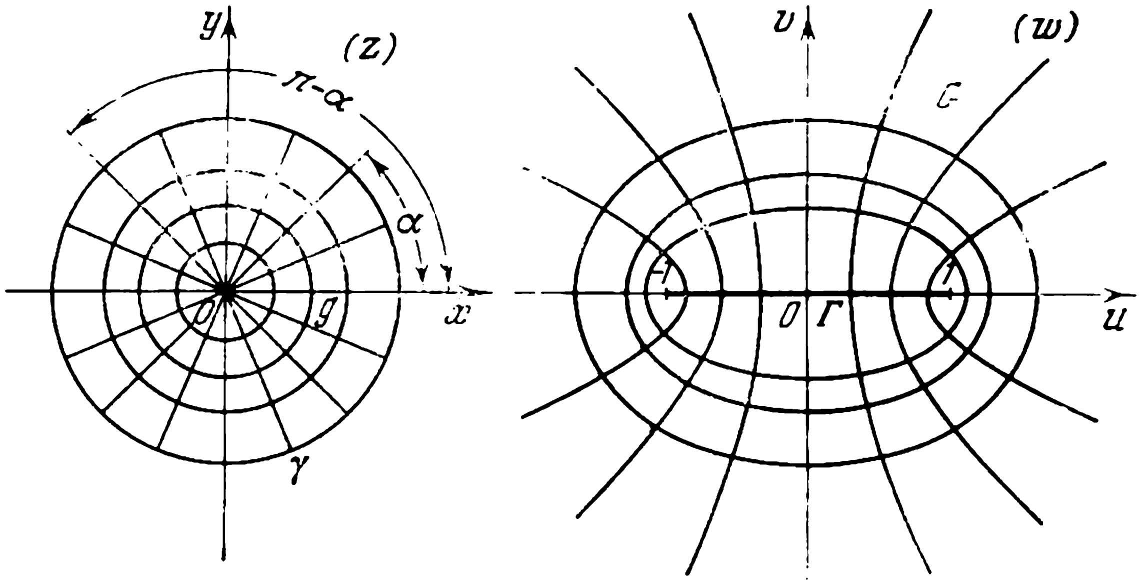

Fig. 14

In order to study the mapping $w = \lambda(z)$ in more detail we find the images for the circles $|z| = r$ and radii $\operatorname{Arg} z = \alpha + 2k\pi$ (Fig. 14). We can restrict ourselves to the interior $|z| < 1$ of the unit circle, for example.

We put

$$z = r(\cos t + i \sin t) \quad (0 \leqslant t \leqslant 2\pi,\ 0 < r < 1).$$

Then

$$w = \frac{1}{2}(z + z^{-1}) = \frac{1}{2}\left(\frac{1}{r} + r\right)\cos t - i\,\frac{1}{2}\left(\frac{1}{r} - r\right)\sin t,$$

or

$$u = \frac{1}{2}\left(\frac{1}{r} + r\right)\cos t, \quad v = -\frac{1}{2}\left(\frac{1}{r} - r\right)\sin t \quad (0 \leqslant t \leqslant 2\pi). \quad (3.18)$$

Solving for t, we obtain

$$\frac{u^2}{\left[\frac{1}{2}\left(\frac{1}{r} + r\right)\right]^2} + \frac{v^2}{\left[\frac{1}{2}\left(\frac{1}{r} - r\right)\right]^2} = 1. \quad (3.19)$$

This is an ellipse with semiaxes $a = \frac{1}{2}\left(\frac{1}{r} + r\right)$ and $b = \frac{1}{2}\left(\frac{1}{r} - r\right)$ and foci ± 1. From (3.18) it follows that when t continuously increases from 0 to 2π (i.e. point z describes the circle $|z| = r$ completely one time in the positive sense), the corresponding point w describes

the ellipse (3.19) one time in the negative sense. Indeed, at $0 \leqslant t \leqslant \pi/2$, u is positive and decreases from a to 0 and v is negative and decreases from 0 to $-b$; at $\pi/2 < t < \pi$, u continues to decrease from 0 to $-a$ while v increases from $-b$ to 0; at $\pi < t < 3\pi/2$, u increases from $-a$ to 0 and v increases from 0 to b; finally, at $3\pi/2 < t < 2\pi$, u increases from 0 to a while v decreases from b to 0.

Varying the radius r of the circle $|z| = r$ from 0 to 1, we make a decrease from ∞ to 1 and b decrease from ∞ to 0; the corresponding ellipses will run through the totality of ellipses in the w-plane with foci ± 1. This implies that the function $w = \lambda(z)$ maps the unit circle in a one-to-one manner into a domain G that is the exterior of a segment Γ of the real axis: $-1 \leqslant x \leqslant 1$. The point at infinity is the image of the center of the unit circle, and the segment Γ (traversed twice) is the image of the unit circle.

For the image of the radius $z = t(\cos\alpha + i\sin\alpha)$ $(0 \leqslant t < 1)$ we obtain first the equation

$$w = \frac{1}{2}\left(\frac{1}{t} + t\right)\cos\alpha - i\frac{1}{2}\left(\frac{1}{t} - t\right)\sin\alpha,$$

or

$$u = \frac{1}{2}\left(\frac{1}{t} + t\right)\cos\alpha, \quad v = -\frac{1}{2}\left(\frac{1}{t} - t\right)\sin\alpha \quad (0 \leqslant t < 1). \tag{3.20}$$

We can see that the images of two radii symmetric with respect to the real axis (if one radius corresponds to angle α, the other corresponds to $-\alpha$) are symmetric with respect to the real axis, and the images of two radii symmetric with respect to the imaginary axis (if one radius corresponds to angle α, the other corresponds to angle $\pi - \alpha$) are symmetric with respect to this axis. For this reason it is sufficient to consider only images of radii belonging to one quadrant, say the first: $0 \leqslant \alpha \leqslant \pi/2$.

Note that for $\alpha = 0$ we have

$$u = \frac{1}{2}\left(\frac{1}{t} + t\right), \quad v = 0 \quad (0 \leqslant t < 1).$$

This is the infinite half-interval $1 < u \leqslant \infty$. The interval symmetric to it, $-\infty \leqslant u < -1$, is the image of a radius with $\alpha = \pi$.

For $\alpha = \pi/2$ we have

$$u = 0, \; v = -\frac{1}{2}\left(\frac{1}{t} - t\right) \quad (0 \leqslant t < 1).$$

This is the imaginary semiaxis $-\infty \leqslant v < 0$. The other imaginary semiaxis, $0 < v \leqslant \infty$, is the image of the radius with $\alpha = -\pi/2$.

Thus, the infinite segment of the real axis starting at -1, passing through the point at infinity ∞, and ending at 1 is the image of the "horizontal" diameter of the unit circle, and the entire imaginary axis with the exception of the origin of coordinate (but including the

point at infinity) is the image of the "vertical" diameter of the unit circle.

Now suppose that $0 < \alpha < \pi/2$. Then, solving Eqs. (3.20) for t, we have

$$\frac{u^2}{\cos^2\alpha} - \frac{v^2}{\sin^2\alpha} = 1. \tag{3.21}$$

This is a hyperbola with the semitransverse axis $a = \cos\alpha$, semiconjugate axis $b = \sin\alpha$, and foci ± 1. Point w, however, does not traverse this hyperbola completely when point z traverses the entire radius $z = t(\cos\alpha + i\sin\alpha)$ $(0 \leqslant t < 1)$. Indeed, from Eqs. (3.20) it follows that when t increases from 0 to 1, u decreases from ∞ to $\cos\alpha$ and v increases from $-\infty$ to 0. Therefore, the point traverses only once a quarter of the hyperbola, the quarter that belongs to the fourth quadrant. In view of the aforesaid, the quarter of the hyperbola in the first quadrant, which is symmetric to the fourth quarter with respect to the real axis, is the image of the radius that is symmetric to the given radius with respect to the real axis, i.e. the radius corresponding to the angle $-\alpha$. But it would be incorrect to say that the hyperbola branches in the first and fourth quadrants are the image of the above-mentioned pair of radii. Indeed, the vertex $u = a$, $v = 0$ of the hyperbola does not belong to this image. (Do not forget that our radii are taken without their end points; a vertex of the hyperbola is the image of these points: $t = 1$.)

On the other hand, the third and second quarters of the hyperbola are the images of the radii corresponding to angles $\pi - \alpha$ and $\alpha + \pi$ (or $\alpha - \pi$).

The complete hyperbola, with the exception of its two vertices, is the image of four radii corresponding to angles $\pm\alpha$ and $\pi \pm \alpha$. Note that for the images of each of the two diameters built from these radii we have the part of the hyperbola made up of two of its quarters symmetric with respect to the origin of coordinates and linked at the point at infinity.

Hence, *the function* $w = \lambda(z) = \frac{1}{2}(z + 1/z)$ *maps in a one-to-one manner both the interior and exterior of the unit circle into the exterior of the segment* $-1 \leqslant u \leqslant +1$ *of the real axis. The circles* $|z| = r$ *are mapped into ellipses with foci* ± 1 *and semiaxes* $\frac{1}{2}(1/r \pm r)$, *and the pairs of diameters symmetric with respect to the coordinate axes (and made up of radii* $z = \pm r(\cos\alpha \pm i\sin\alpha)$ $(0 \leqslant r < 1)$) *are mapped into hyperbolas with foci* ± 1 *and semiaxes* $|\cos\alpha|$ *and* $|\sin\alpha|$, *with the exception of their vertices.*

Since the derivative

$$w = \lambda'(z) = \frac{1}{2}\left(1 - \frac{1}{z^2}\right)$$

is nonzero for $z \neq \pm 1$, the mapping is conformal at all points of the domains under consideration (the interior and exterior of the unit circle). It follows then that the hyperbolas intersect the ellipses at angles at which the radii intersect the circles, i.e. right angles.

Let us consider other images of circles passing through points ± 1. From

$$w = \frac{1}{2}\left(z + \frac{1}{z}\right) = \lambda(z)$$

we find that

$$w - 1 = \frac{z^2 - 2z + 1}{2z} = \frac{(z-1)^2}{2z}, \quad w + 1 = \frac{z^2 + 2z + 1}{2z} = \frac{(z+1)^2}{2z},$$

whence

$$\frac{w-1}{w+1} = \left(\frac{z-1}{z+1}\right)^2.$$

It is easy to see that this equation is equivalent to the given equation. Assuming that $(z - 1)/(z + 1) = z'$ and $(w - 1)/(w + 1) = w'$,

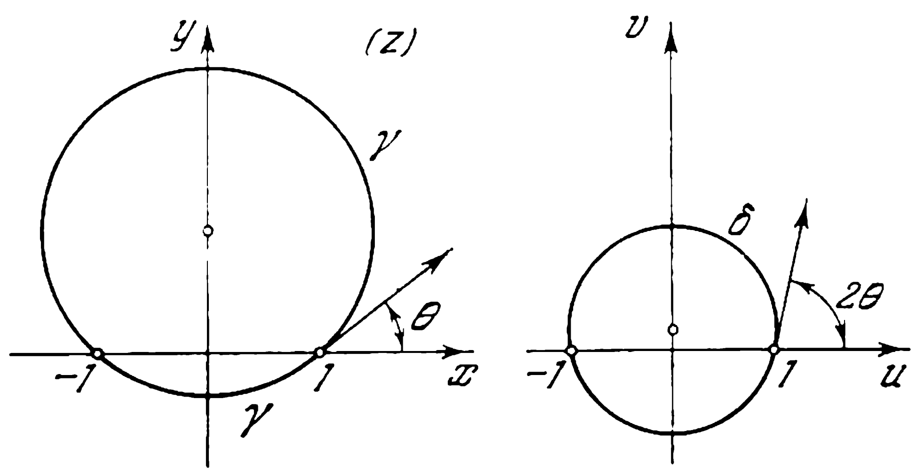

Fig. 15

we find that the mapping $w = \lambda(z)$ can be changed by the following:

$$z' = \frac{z-1}{z+1}, \quad w' = z'^2, \text{ and } \frac{w-1}{w+1} = w'. \tag{3.22}$$

The first function maps a circle passing through point ± 1 into straight lines passing through the origin of coordinates, the second maps each of these straight lines into a ray emerging from the origin of coordinates, and the last maps each ray into an arc of a circle connecting points -1 and $+1$. From (3.22) it follows that if at point $z = 1$ the angle between a circle and the positive direction of the real axis is θ, the angle at point $w = 1$ between its image (an arc of a circle) and the positive direction of the real axis is 2θ (Fig. 15).

Therefore, the function $w = \lambda(z)$ maps each circle γ passing through points ± 1 and making at point 1 an angle θ with the positive direction of the real axis into an arc δ of a circle passing through points ± 1 and making an angle 2θ with the positive direction of the

real axis. The same formula (3.22) show that as a result of mapping each of the two arcs γ with end points at ± 1 is carried separately into one and the same arc δ.

Note that the exterior of circle γ is carried under the first mapping in (3.22) into a half-plane, under the second mapping into a domain bounded by a ray starting at the origin of coordinates, and under the third mapping into a domain whose boundary is the arc δ. Since all these mappings are one-to-one in the corresponding domains, the

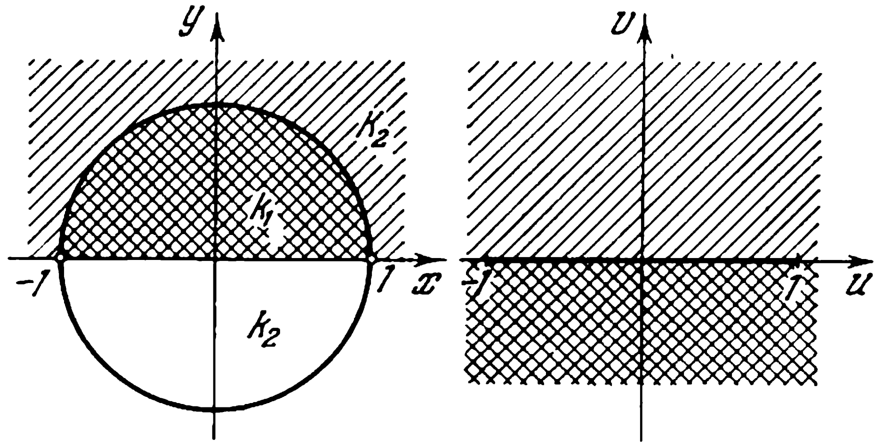

Fig. 16

function $w = \lambda(z)$ provides a one-to-one and conformal mapping of the exterior of circle γ (and also of the interior of this circle) into a domain whose boundary is the arc of a circle δ connecting points -1 and $+1$.

It is important to note that the function $w = \lambda(z)$ maps the half-disk $k_1 : |z| < 1$ in the upper half-plane into the lower half-plane of w and the half-disk k_2 in the lower half-plane into the upper half-plane. But since $\lambda(z) = \lambda(1/z)$, at each point in the half-disk k_2 the function takes on the same values as at the points of the upper half-plane external to the half-disk k_1. If we denote the set of latter points by K_2 (Fig. 16), we can state that the upper half-plane is also the image of domain K_2. Finally, we take into account the fact that the once-traversed segment $-1 < u < 1$ is the image of the semicircle separating k_1 and K_2. This means that under the mapping $w = \lambda(z)$ the upper half-plane is carried into a domain that consists of the upper and lower half-planes and the segment $-1 < u < 1$ of the real axis, i.e. the entire w-plane except the segment of the real axis starting at -1, passing through the point at infinity, and ending at 1.

3.11

THE EXPONENTIAL FUNCTION

The exponential function e^x of the real variable x can be defined as the solution of the differential equation $df/dx = f$ with the initial condition $f(0) = 1$. We wish to find an analytic function $f(z) =$

$= u(x, y) + iv(x, y)$ satisfying similar criteria:

$$\frac{df}{dz} = f \quad \text{and} \quad f(0) = u(0,\ 0) + iv(0,\ 0) = 1.$$

Since $\frac{df}{dz} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$, the differential equation assumes the form

$$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y} = u + iv,$$

whence

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = u, \qquad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = v.$$

Starting from the differential equation $\partial u/\partial x = u$ and the initial condition $u(0, 0) = 1$, we find that $u(x, y) = e^x\lambda(y)$, where $\lambda(y)$ is a differentiable function such that $\lambda(0) = 1$. Similarly, starting from the equation $\partial v/\partial x = v$ and the initial condition $v(0, 0) = 0$, we find that $v(x, y) = e^x\mu(y)$, where $\mu(y)$ is a differentiable function satisfying the condition $\mu(0) = 0$. Therefore,

$$\frac{\partial v}{\partial y} = e^x\mu'(y) = u = e^x\lambda(y), \quad -\frac{\partial u}{\partial y} = -e^x\lambda'(y) = v = e^x\mu(y),$$

i.e.

$$\mu'(y) = \lambda(y), \quad -\lambda'(y) = \mu(y).$$

Obviously, $\lambda(y)$ and $\mu(y)$ obey the same second-order linear differential equation

$$\frac{d^2\varphi(y)}{dy^2} + \varphi(y) = 0,$$

whose general solution is $\varphi(y) = C_1 \cos y + C_2 \sin y$. Allowing for the initial condition for $\mu(y)$, we find that $\mu(y) = C \sin y$ and, hence, $\lambda(y) = \mu'(y) = C \cos y$, wherefrom $C = 1$, in view of the initial condition for $\lambda(y)$.

Thus, $\lambda(y) = \cos y$, $\mu(y) = \sin y$, and

$$f(z) = u(x, y) + iv(x, y) = e^x(\cos y + i \sin y).$$

We have found the only analytic function (analytic in the entire plane, i.e. an entire function) that satisfies the above criteria. This is known as the *exponential function of a complex variable* and is denoted by exp z:

$$\exp z = e^x(\cos y + i \sin y).$$

Note that for z real ($y = 0$ and $z = x$) we arrive at

$$\exp x = e^x,$$

i.e. on the real axis the exponential function of a complex variable coincides with that of a real variable (the latter is known from the general calculus course).

Moreover, we can verify directly that the *addition theorem* is true for exp z and repeats the respective addition theorem for e^x, i.e. $e^{x_1}e^{x_2} = e^{x_1+x_2}$; namely, $\exp_{z_1} \cdot \exp_{z_2} = \exp(z_1 + z_2)$.

We will call the complex number z the *exponent* in exp z. The addition theorem can be formulated thus: *when exponential functions are multiplied*, *their exponents are added*. This justifies the use of the notation e^z along with exp z:

$$e^z = e^x (\cos y + i \sin y).$$

3.12
MAPPING BY THE EXPONENTIAL FUNCTION

From the definition of the exponential function

$$\exp z = e^x (\cos y + i \sin y) \tag{3.23}$$

it follows that it is nonzero for all values of z and that

$$| \exp z | = e^x \text{ and } \operatorname{Arg} (\exp z) = y + 2k\pi.$$

At $z = iy$ $(x = 0)$ we find that

$$\exp (iy) = \cos y + i \sin y.$$

This relationship enables us to replace the trigonometric form of notation for complex numbers

$$c = r (\cos \varphi + i \sin \varphi) \quad (r \neq 0)$$

with the more compact exponential form

$$c = r \exp (i\varphi) = re^{i\varphi}.$$

Formula (3.23) shows that the exponential function has a period of $2\pi i$ (since when y changes by 2π the function z changes by $2\pi i$ but the value of the function remains unchanged):

$$\exp (z + 2\pi i) = \exp z.$$

Let us show that $2\pi i$ is the *basic period* of the exponential function, i.e. any other period is of the type $2\pi ki$, where k is an integer.

Indeed, suppose that $\omega = \alpha + i\beta$ is a period of the exponential function. Then

$$\exp (z + \omega) \exp z$$

for all values of z and, for instance, $z = 0$:

$$\exp \omega = \exp (\alpha + i\beta) = e^\alpha (\cos \beta + i \sin \beta) = 1.$$

But this means that $e^{\alpha} = 1$, i.e. $\alpha = 0$, and $\cos \beta + i \sin \beta = 1$, i.e. $\beta = 2k\pi$. Hence

$$\omega = \alpha + i\beta = 2k\pi i,$$

which we set out to prove.

We will consider the expression $\exp \infty$ meaningless, since $\lim\limits_{z \to \infty} e^z$ does not exist. It suffices to note that $e^x \to \infty$ for positive x's tending to ∞, while $e^x \to 0$ for negative x's tending to $-\infty$.

From this it follows, for one, that $\exp z$ cannot be represented by a polynomial of any order, since a polynomial that is not a constant tends to ∞ as $z \to \infty$. Entire functions different from polynomials are called *transcendental entire functions*, which means that $\exp z$ is a transcendental entire function.

Since by definition

$$(\exp z)' = \exp z,$$

the derivative of the exponential function is nonzero for all values of z.

Let us investigate the geometrical behavior of $w = \exp z$, i.e. the mapping that this function carries out. We have noted above that the value $w = 0$ is not assumed by this function anywhere. This means that the origin of coordinates does not belong to the image of the finite z-plane under the mapping $w = \exp z$. We wish to show that any other finite point in the w-plane does belong to this image. Indeed, the equation $w = \exp z$ with $w \neq 0$ given and $z = x + iy$ the unknown yields

$$| w | = e^x, \text{ whence } x = \ln | w |$$

and

$$\operatorname{Arg} w = y + 2\pi k, \text{ i.e. } y = \operatorname{Arg} w.$$

Therefore only the points

$$z = \ln | w | + i \operatorname{Arg} w$$

can be the preimages of points w. Obviously there is an infinite number of such points, since $\operatorname{Arg} w$ has an infinite number of values that differ pairwise by an integral multiple of 2π. Moreover, each of these points is indeed a preimage of w, since

$$\begin{aligned}\exp (\ln | w | + i \operatorname{Arg} w) &= e^{\ln |w|} (\cos \operatorname{Arg} w + i \sin \operatorname{Arg} w) \\ &= | w | (\cos \operatorname{Arg} w + i \sin \operatorname{Arg} w) = w.\end{aligned}$$

Hence, *the set of all roots of the equation* $w = e^z$ $(w \neq 0)$ *is represented by the function*

$$z = \ln | w| + i \operatorname{Arg} w = \ln |w | + i (\arg w + 2k\pi), \quad (3.24)$$

where $k = 0, \pm 1, \pm 2, \ldots$. All these points are on one straight line parallel to the imaginary axis and are 2π apart.

We have discovered that the function $w = \exp z$ maps the finite z-plane into a domain that consists of the finite w-plane without point $w = 0$, and the mapping is not one-to-one because each point $w \neq 0$ has an infinite number of preimages (3.24).

Since the derivative of the exponential function is everywhere nonzero, this mapping is conformal in the finite z-plane.

Let us make z traverse a straight line parallel to one of the coordinate axes (Fig. 17). If this is the straight line $z = c + it$, parallel to the imaginary axis, then $w = e^c (\cos t + i \sin t)$, i.e. w is on a circle with its center at the origin of coordinates and a radius equal to e^c. As point z once traverses the straight line so that the ordinate of this point, t, grows from $-\infty$ to $+\infty$, point w traverses the corresponding circle an infinite number of times in one direction (positive).

But if the point z traverses a straight line $z = t + ic'$, parallel to the real axis, then $w = e^t (\cos c' + i \sin c')$ traverses a straight ray starting at the origin of coordinates and making an angle c' with the positive direction of the real axis. As point z once traverses the straight line so that the abscissa of this point, t, grows from $-\infty$ to $+\infty$, point w once traverses the corresponding ray so that the distance between this point and the origin of coordinates grows from 0 to ∞ (both end points are, of course, excluded since $|w| = e^t$).

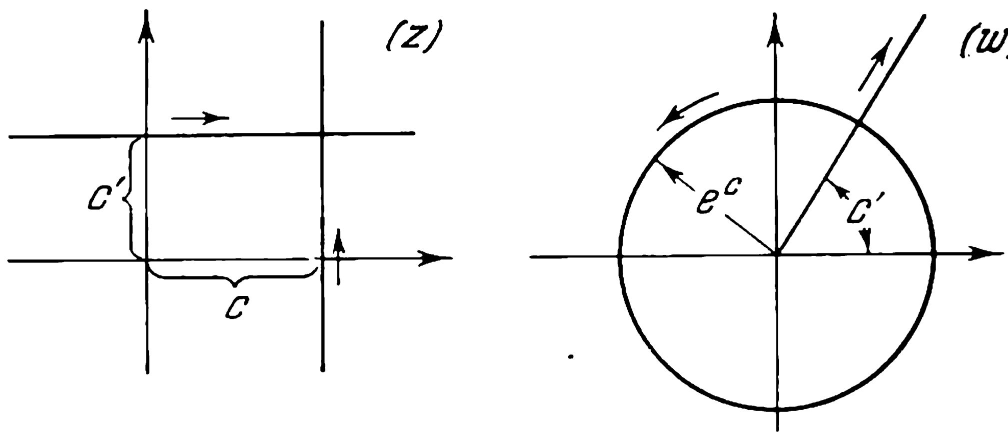

Fig. 17

Hence, *under the mapping $w = e^z$ of the z-plane, the family of straight lines parallel to the imaginary axis is carried into the family of circles with their centers at the origin of coordinates, and the family of straight lines parallel to the real axis is carried into the family of rectilinear rays starting at the origin of coordinates.*

Let us consider a domain g that is the interior of a straight strip of width h $(0 < h \leqslant 2\pi)$ parallel to the real axis. We assume that this strip is bounded by the straight lines $y = \varphi_0$ and $y = \varphi_1$ $(\varphi_1 - \varphi_0 = h)$. It follows then that in the w-plane a domain d consisting of the interior of an angle with the opening span h, the vertex at the origin of coordinates, and bounded by rectilinear rays $\mathrm{Arg}\, w = \varphi_0 +$

$+ 2k\pi$ and $\operatorname{Arg} w = \varphi_1 + 2l\pi$ (Fig. 18) is the image of g. The correspondence between points of d and g established by the mapping $w = \exp z$ is one-to-one. To verify the last statement it is enough to note that the preimages of a point $w \in d$ may be only points $\ln |w| + i \operatorname{Arg} w$, which differ only in their imaginary parts. Two such points lie on a straight line parallel to the imaginary axis and are separated by a distance that is an integral multiple of 2π. But since our strip g has a width not greater than 2π, it cannot contain more than one preimage of the point w. Hence, each point $z \in g$ has only one image point and each point $w \in d$ has only one preimage point, a fact which proves that the mapping is one-to-one.

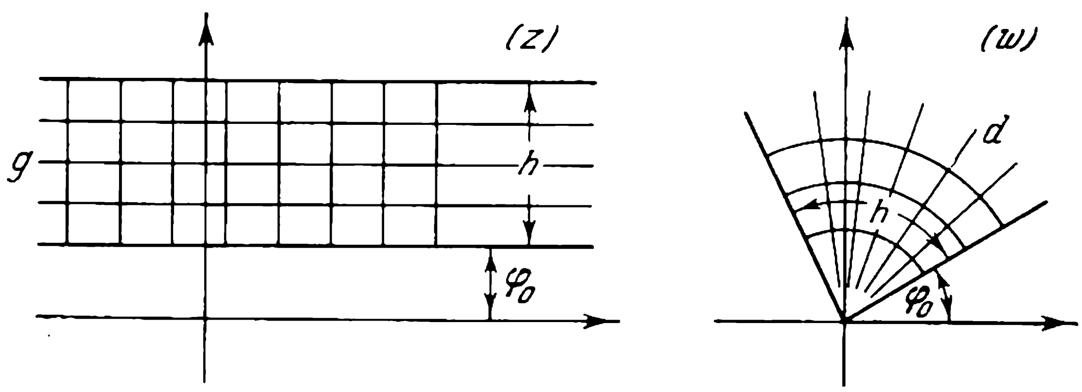

Fig. 18

We see that *the exponential function $w = \exp z$ maps a strip of width $h \leqslant 2\pi$ parallel to the real axis into the interior of an angle with the opening span h and the vertex at the origin of coordinates in a one-to-one manner and conformally.*

For this reason the exponential function is used every time that a strip must be mapped conformally into the interior of an angle.

If we take a straight line in the z-plane that is not parallel to any of the two coordinate axes, its image in the w-plane is neither a straight line nor a circle but the logarithmic spiral. Indeed, if the straight line is

$$z = t(1 + i\alpha) + bi \quad (-\infty < t < +\infty)$$

where α is the slope and b the ordinate at the origin of coordinates, then its image is

$$w = \exp[t + i(\alpha t + b)] = e^t [\cos(\alpha t + b) + i \sin(\alpha t + b)].$$

Here

$$|w| = r = e^t, \quad \varphi = \operatorname{Arg} w = \alpha t + b + 2m\pi,$$

or, after solving for t, $r = \exp \dfrac{\varphi - b - 2m\pi}{\alpha}$. But $\operatorname{Arg} w$ (the polar angle φ) is defined only to within an integral multiple of 2π. Con-

sequently, by denoting $\varphi - 2m\pi$ again by φ, we obtain

$$r = ce^{\frac{\varphi}{\alpha}}, \quad \text{where} \quad c = e^{-\frac{b}{\alpha}}.$$

This is the *logarithmic spiral* (Fig. 19). In view of the fact that this is the image of the straight line $z = t(1 + i\alpha) + ib$, which intersects straight lines parallel to the real axis at an angle arc tan α, and that the mapping is conformal, the logarithmic spiral intersects the images of these straight lines at the same (constant) angle. (We recall

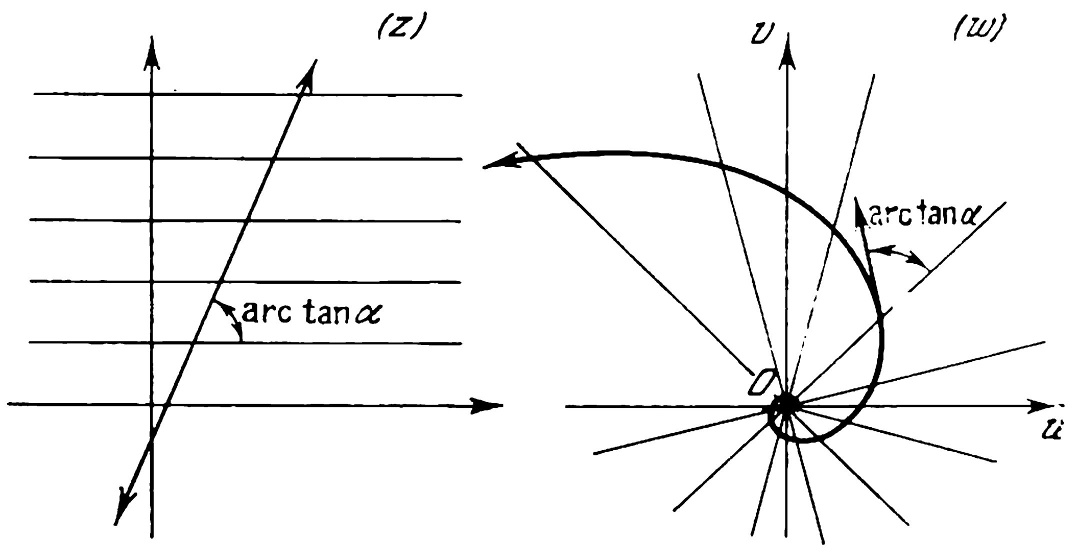

Fig. 19

that the images of these straight lines are rays emerging from the origin of coordinates.) We have the characteristic feature of the logarithmic spiral.

There is a similarity between the mappings $w = (z - a)^n$ and $w = \exp z$. We can establish it by means of the formula

$$\exp z = \lim_{n\to\infty} \left(1 + \frac{z}{n}\right)^n,$$

the proof of which we leave to the reader.*

Consider the mapping $w = \left(1 + \frac{z}{n}\right)^n = \frac{1}{n^n}\left[z - (-n)\right]^n$, with respect to which the mapping $w = \exp z$ is the limit. In view of the results of Sec. 3.3, this mapping carries an angle with an opening span h/n $(0 < h \leqslant 2\pi)$, the vertex at point A_n $(-n, 0)$, and bounded by a section of the real axis $x \geqslant -n$ $(y = 0)$ and the ray $\operatorname{Arg}(z + n) = h/n + 2k\pi$ into an angle with an opening span h, the vertex at the origin of coordinates, and bounded by the rays

* A proof of this formula is given in A. I. Markushevich and L. A. Markushevich, *Introduction to Analytic Function Theory* [in Russian], Prosveshchenie, Moscow, 1977, p. 234.

Arg $w = 0$ and Arg $w = h + 2m\pi$ (Fig. 20). If n tends to infinity, the vertex at A_n goes to infinity along the negative part of the real axis and the length of OB_n tends to the limit $\lim_{n\to\infty} n \tan(h/n) = h$ in a way such that the limiting position of ray A_nB_n coincides with the straight line $y = h$, which together with the real axis are the

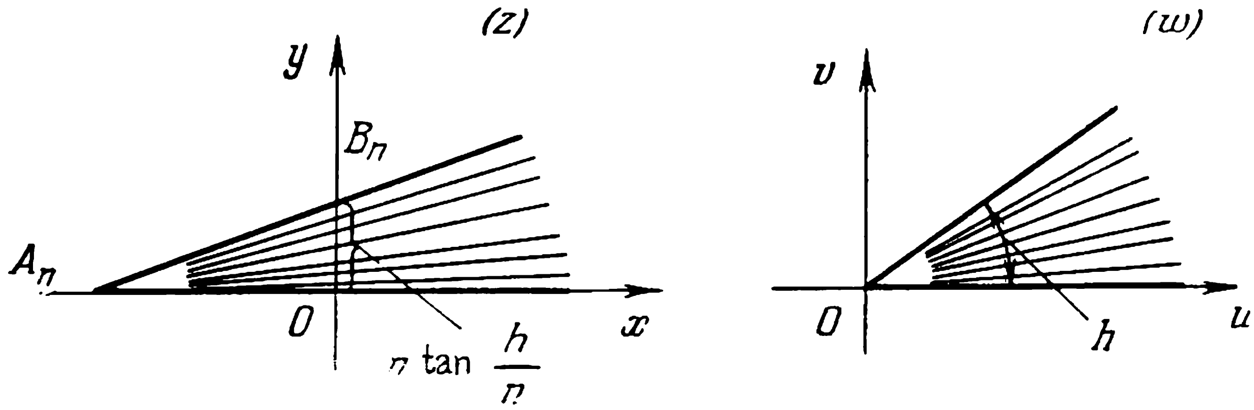

Fig. 20

boundaries of a strip of width h. It is obvious that the limiting position of rays emerging from the origin of coordinates will be the straight lines parallel to the real axis, and the limiting position of arcs of circles with centers at point A_n will be the segments of straight lines perpendicular to the real axis and inside the strip. We see that mapping by means of the exponential function can be obtained from mapping by means of a power function by an appropriate limiting transition.

3.13

TRIGONOMETRIC FUNCTIONS

Let us turn to the definition of the sine and cosine of a complex number. From

$$\exp(ix) = \cos x + i \sin x \text{ and } \exp(-ix) = \cos x - i \sin x$$

we obtain the well-known Euler formulas:

$$\cos x = \frac{\exp(ix) + \exp(-ix)}{2}, \quad \sin x = \frac{\exp(ix) - \exp(-ix)}{2i},$$

which are therefore valid for all real x's. Since the right-hand sides of these formulas are valid for all complex numbers $z \neq \infty$ and are therefore analytic functions of z, we have two entire functions of z,

$$\frac{\exp(iz) + \exp(-iz)}{2} \text{ and } \frac{\exp(iz) - \exp(-iz)}{2i},$$

that for real values $z = x$ take on real values corresponding to $\cos x$ and $\sin x$, respectively. By definition, the first is denoted by $\cos z$

and is called the *cosine* z while the second is denoted by $\sin z$ and is called the *sine* z:

$$\left.\begin{aligned}\cos z&=\frac{\exp(iz)+\exp(-iz)}{2},\\ \sin z&=\frac{\exp(iz)-\exp(-iz)}{2i}.\end{aligned}\right\}\tag{3.25}$$

These are the main *trigonometric functions*. Formulas (3.25) are also called the *Euler formulas*, and so are the formulas obtained by termwise multiplication of the second row in (3.25) by i and addition of the result to the first row:

$$\exp(iz) = \cos z + i \sin z. \tag{3.26}$$

From (3.25) it follows directly that $\cos z$ is an *even* function and $\sin z$ an *odd* function:

$$\left.\begin{aligned}\cos(-z)&=\cos z,\\ \sin(-z)&=-\sin z.\end{aligned}\right\}\tag{3.27}$$

From (3.25) we can also see that both $\cos z$ and $\sin z$ are periodic functions with a period of 2π (since as z changes by 2π the iz and $-iz$ on the right-hand sides of (3.25) change by $2\pi i$, the period of the exponential function). We wish to show that 2π is the *basic period* of $\cos z$ and $\sin z$. True, since if ω is a period of $\cos z$, we find that

$$\cos(z+\omega) = \cos z,$$

which for $z = \pi/2$ becomes

$$\cos\left(\omega+\frac{\pi}{2}\right)=0.$$

But this implies that

$$\exp\left[i\left(\omega+\frac{\pi}{2}\right)\right]+\exp\left[-i\left(\omega+\frac{\pi}{2}\right)\right]=0,$$

or

$$\exp[i(2\omega+\pi)] = -1.$$

Hence, by (3.24) we have

$$i(2\omega+\pi) = \ln|-1| + i\operatorname{Arg}(-1) = i(\pi+2k\pi),$$

i.e. $\omega = k\pi$, and since $\cos\omega = \cos 0 = 1$, we find that k is an even number and

$$\omega = 2m\pi.$$

In the same manner we can establish that 2π is the basic period of $\sin z$.

Let us now turn to the *addition theorem* for $\cos z$ and $\sin z$, i.e. the relationships that connect $\cos(z_1+z_2)$ and $\sin(z_1+z_2)$ on the

one hand with $\cos z_1, \cos z_2, \sin z_1$, and $\sin z_2$ on the other (z_1 and z_2 being arbitrary complex numbers). We find these relationships as corollaries of the addition theorem for the exponential function.

Substituting $z_1 + z_2$ for z in (3.26), we find that

$$\cos(z_1 + z_2) + i\sin(z_1 + z_2) = \exp[i(z_1 + z_2)] = \exp(iz_1)\cdot\exp(iz_2)$$
$$= (\cos z_1 + i\sin z_1)(\cos z_2 + i\sin z_2)$$

or, after carrying out the multiplication,

$$\cos(z_1 + z_2) + i\sin(z_1 + z_2)$$
$$= (\cos z_1 \cos z_2 - \sin z_1 \sin z_2) + i(\sin z_1 \cos z_2 + \cos z_1 \sin z_2).$$

Substituting $-z_1$ for z_1 and $-z_2$ for z_2 and using (3.27), we obtain

$$\cos(z_1 + z_2) - i\sin(z_1 + z_2)$$
$$= (\cos z_1 \cos z_2 - \sin z_1 \sin z_2) - i(\sin z_1 \cos z_2 + \cos z_1 \sin z_2).$$

Adding and subtracting the two last formulas termwise, we find that

$$\left.\begin{aligned}\cos(z_1 + z_2) &= \cos z_1 \cos z_2 - \sin z_1 \sin z_2,\\ \sin(z_1 + z_2) &= \sin z_1 \cos z_2 + \cos z_1 \sin z_2.\end{aligned}\right\} \qquad (3.28)$$

These are the main formulas in the theory of trigonometric functions. They include as a particular case very important formulas from the practical viewpoint. If in (3.28) we put $z_1 = z$ and $z_2 = \pi/2$, then

$$\cos\left(z + \frac{\pi}{2}\right) = \cos z \cos\frac{\pi}{2} - \sin z \sin\frac{\pi}{2} = -\sin z,$$
$$\sin\left(z + \frac{\pi}{2}\right) = \sin z \cos\frac{\pi}{2} + \cos z \sin\frac{\pi}{2} = \cos z.$$

On the other hand, if we put $z_1 = z$ and $z_2 = \pi$, we arrive at another important case:

$$\cos(z + \pi) = -\cos z,$$
$$\sin(z + \pi) = -\sin z.$$

This process can be continued for $z_2 = 3\pi/2$ and $z_2 = 2\pi$.

If in the first formula in (3.28) we put $z_1 = z$ and $z_2 = -z$, we arrive at the following relationship between $\cos z$ and $\sin z$:

$$1 = \cos^2 z + \sin^2 z. \qquad (3.29)$$

We see that all known formulas from trigonometry are carried over without change to the complex plane. But here (3.29) does not imply that $|\cos z| \leqslant 1$ or $|\sin z| \leqslant 1$, since generally speaking neither $\cos^2 z$ nor $\sin^2 z$ is a nonnegative real number.

Other functions closely linked with $\sin z$ and $\cos z$ are the *hyperbolic functions* $\cosh z$ and $\sinh z$ defined as

$$\cosh z = \frac{\exp z + \exp(-z)}{2}, \quad \sinh z = \frac{\exp z - \exp(-z)}{2}. \tag{3.30}$$

For real values x of z these functions assume real values and coincide with the functions $\cosh x$ and $\sinh x$ known from the calculus course. The first real-valued function of x (an even function) decreases in the interval $-\infty < x \leqslant 0$ from ∞ to 1 and then increases from 1 to ∞ in the interval $0 \leqslant x < +\infty$. The second function (an odd function) increases in the entire interval $-\infty < x < +\infty$ from $-\infty$ to $+\infty$ and passes through zero at $x = 0$.

Comparing (3.30) with (3.25) we find the following formulas linking trigonometric and hyperbolic functions:

$$\cosh z = \cos(iz), \quad \sinh z = -i \sin(iz). \tag{3.31}$$

This, for one, implies that

$$\cosh^2 z - \sinh^2 z = [\cos(iz)]^2 + [\sin(iz)]^2 = 1. \tag{3.32}$$

Let us determine the real and imaginary parts and the moduli of $\cos z$ and $\sin z$. Assuming that $z = x + iy$, by (3.28) and (3.31) we find that

$$\left.\begin{aligned} \cos(x+iy) &= \cos x \cos(iy) - \sin x \sin(iy) \\ &= \cos x \cosh y - i \sin x \sinh y, \\ \sin(x+iy) &= \sin x \cos(iy) + \cos x \sin(iy) \\ &= \sin x \cosh y + i \cos x \sinh y. \end{aligned}\right\} \tag{3.33}$$

Whence

$$\left.\begin{aligned} \operatorname{Re}\cos(x+iy) &= \cos x \cosh y, \\ \operatorname{Im}\cos(x+iy) &= -\sin x \sinh y, \\ \operatorname{Re}\sin(x+iy) &= \sin x \cosh y, \\ \operatorname{Im}\sin(x+iy) &= \cos x \sinh y. \end{aligned}\right\} \tag{3.34}$$

For the moduli of $\cos z$ and $\sin z$ we arrive at the following expressions

$$\begin{aligned} |\cos z| &= \underset{+}{\sqrt{}}\,\overline{(\cos x \cosh y)^2 + (\sin x \sinh y)^2} \\ &= \underset{+}{\sqrt{}}\,\overline{\cosh^2 y\,(1 - \sin^2 x) + \sin^2 x \sinh^2 y} = \underset{+}{\sqrt{}}\,\overline{\cosh^2 y - \sin^2 x} \end{aligned}$$

and similarly, $|\sin z| = \underset{+}{\sqrt{}}\,\overline{\sinh^2 y + \sin^2 x}$. Thus

$$\left.\begin{aligned} |\cos z| &= \underset{+}{\sqrt{}}\,\overline{\cosh^2 y - \sin^2 x}, \\ |\sin z| &= \underset{+}{\sqrt{}}\,\overline{\sinh^2 y + \sin^2 x}. \end{aligned}\right\} \tag{3.35}$$

This brings us to the following inequalities:

$$\left.\begin{aligned} \cosh y \geqslant |\cos z| \geqslant \underset{+}{\sqrt{\cosh^2 y - 1}} = |\sinh y|, \\ \underset{+}{\sqrt{\sinh^2 y + 1}} = \cosh y \geqslant |\sin z| \geqslant |\sinh y|. \end{aligned}\right\} \qquad (3.36)$$

On the other hand, the inequalities (3.36) can be found directly from (3.25). For instance,

$$|\cos z| \leqslant \frac{|\exp(iz)| + |\exp(-iz)|}{2} = \frac{\exp(-y) + \exp y}{2} = \cosh y,$$

$$|\cos z| \geqslant \left| \frac{|\exp(iz)| - |\exp(-iz)|}{2} \right| = \left| \frac{\exp(-y) - \exp y}{2} \right| = \sinh y|,$$

from which we conclude that

$$|\cos z| \to \frac{1}{2}\exp|y| \quad \text{and} \quad |\sin z| \to \frac{1}{2}\exp|y|.$$

as $|y| \to \infty$.

Figure 21 presents the surface $u = |\sin z|$, the modular surface of $\sin z$. Since $\sinh y \neq 0$ at $y \neq 0$, it follows from (3.36) that neither

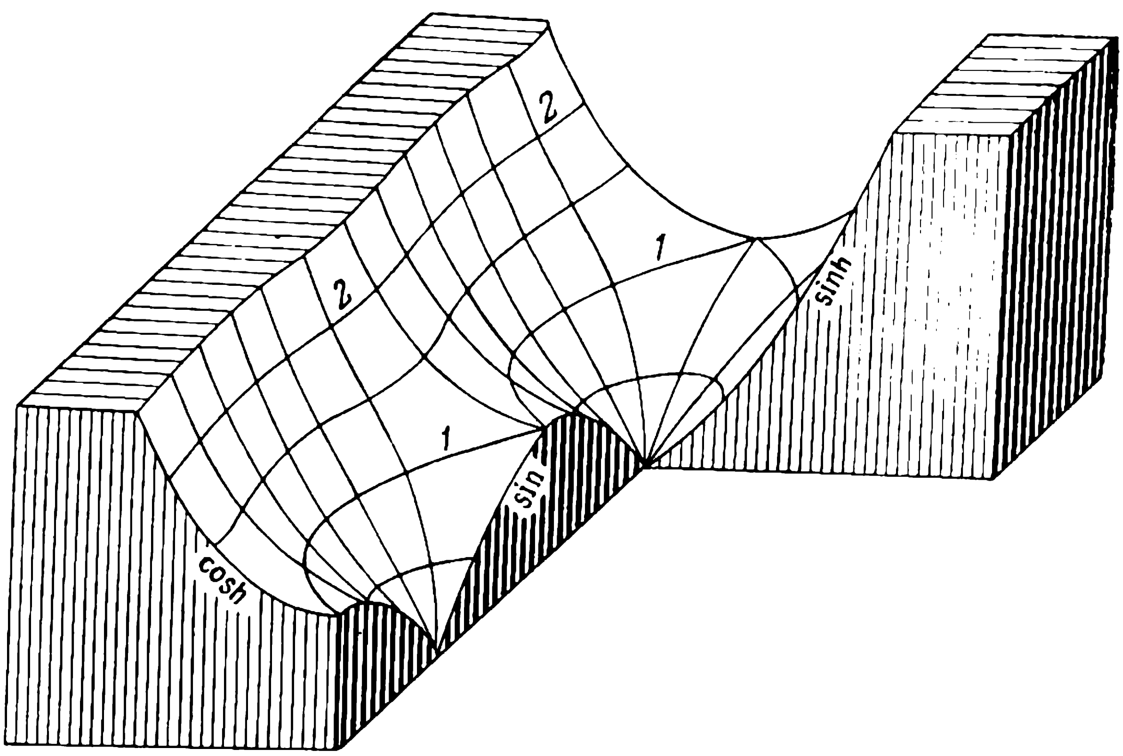

Fig. 21

$\cos z$ nor $\sin z$ vanishes outside the real axis, i.e. the equations $\sin z = 0$ and $\cos z = 0$ have no imaginary roots. Therefore, all roots of these equations are known from elementary trigonometry; namely,

$$z = (2k - 1)\frac{\pi}{2} \text{ for the equation } \cos z = 0$$

and

$$z = k\pi \text{ for the equation } \sin z = 0.$$

Note some formulas for derivatives of trigonometric and hyperbolic functions:

$$(\cos z)' = \left[\frac{\exp(iz) + \exp(-iz)}{2}\right]' = i\,\frac{\exp(iz) - \exp(-iz)}{2} = -\sin z,$$

$$(\sin z)' = \cos z, \quad (\cosh z)' = \sinh z, \quad (\sinh z)' = \cosh z.$$

Other trigonometric functions of a complex variable can be determined from $\sin z$ and $\cos z$:

$$\tan z = \frac{\sin z}{\cos z}, \quad \cot z = \frac{\cos z}{\sin z}, \quad \sec z = \frac{1}{\cos z}, \quad \csc z = \frac{1}{\sin z}.$$

3.14

THE GEOMETRIC BEHAVIOR OF TRIGONOMETRIC FUNCTIONS I

Let us study the geometric behavior of trigonometric functions. We can restrict ourselves by the mapping

$$w = \cos z,$$

since the mapping $w = \sin z$ can be represented as

$$w = -\cos\left(z + \frac{\pi}{2}\right)$$

and, therefore, is reduced to the translation $z_1 = z + \pi/2$ of the z-plane along the real axis, the mapping $z_2 = \cos z_1$, and, finally, the rotation of the z-plane around the origin of coordinates by an angle π, i.e. $w = -z_2$.

We start with the preimages of point w in the mapping $w = \cos z$, i.e. the roots of the equation

$$w = \cos z, \tag{3.37}$$

where w is an arbitrary complex number but not the point at infinity. Using the formula for $\cos z$ from (3.25) and assuming, for brevity, that

$$\exp(iz) = t, \tag{3.38}$$

we arrive at the following equation for t:

$$w = \frac{t + t^{-1}}{2},$$

or

$$t^2 - 2wt + 1 = 0, \tag{3.39}$$

whence

$$t_j = w + \sqrt{w^2 - 1} \quad (j = 1, 2) \tag{3.40}$$

(there is no "plus-minus" sign in front of the square root because this root has two values as it is). The product of t_1 and t_2 is unity, obviously; then both are nonzero. By denoting one of them by τ and the other by $1/\tau$, from (3.38) we find two equations for z:

$$\exp(iz) = \tau\ (\neq 0) \text{ and } \exp(iz) = \frac{1}{\tau}\ (\neq 0). \tag{3.41}$$

According to Sec. 3.12 each of these equations has an infinitude of solutions expressed by (3.24):

$$iz' = \ln|\tau| + i \operatorname{Arg} \tau \text{ and } iz'' = \ln\left|\frac{1}{\tau}\right| + i \operatorname{Arg}\frac{1}{\tau} = -(\ln|\tau| + i \operatorname{Arg}\tau),$$

or

$$z' = \operatorname{Arg}\tau - i\ln|\tau| \text{ and } z'' = -(\operatorname{Arg}\tau - i\ln|\tau|). \tag{3.42}$$

We have arrived at two infinite sets of points on a pair of straight lines $y = \pm\ln|\tau|$ parallel to the real axis. On each straight line

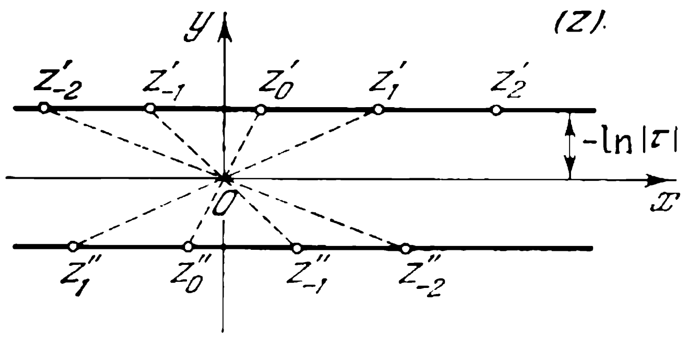

Fig. 22

the adjacent points z' (on one line), or z'' (on the other), stand apart by a distance of 2π. Moreover, for each point z' on the straight line $y = -\ln|\tau|$ there is a point z'' on $y = \ln|\tau|$ symmetric to z' with respect to the origin of coordinates (Fig. 22 corresponds to the case where $|\tau| < 1$). At $\omega = \pm 1$ the roots τ and $1/\tau$ of Eq. (3.39) become equal to ± 1. Then the two straight lines merge with the real axis and so do the two sets of points z' and z''.

Thus, Eq. (3.37) always has a solution and the set of solutions is always infinite. This implies, first, that the function $w = \cos z$ maps the finite z-plane into the entire finite w-plane and, second, that each point w has an infinite number of preimages in the z-plane. The mapping is conformal at all points for which $(\cos z)' = -\sin z \neq 0$, i.e. for $z \neq k\pi$ $(k = 0, \pm 1, \pm 2, \ldots)$.

Suppose that z traverses a straight line that is parallel to one of the coordinate axes. If it is $z = c + it$, which is parallel to the imaginary axis, the image curve will be L: $w = \cos z = \cos c \cosh t - i \sin c \sinh t$ (see the first formula in (3.33)). At $c = k\pi$ we have $w = \cos z = \cos k\pi \cosh t = (-1)^k \cosh t\ (-\infty < t < +\infty)$, i.e. w twice traverses the section $u \geqslant 1$ of the real axis for k even

and the section $u \leqslant -1$ for k odd. At $c = (2k-1)\pi/2$ we have $w = (-1)^k i \sinh t$, i.e. w once traverses the entire imaginary axis in the direction of increasing v's for k even and in the direction of decreasing v's for k odd.

Now we assume that $c \neq m\pi/2$ (for any integer m). We write the equation for curve L in the form

$$u = \cos c \cosh t, \quad v = -\sin c \sinh t \quad (-\infty < t < \infty), \tag{3.43}$$

or, solving for parameter t ($\cos c \neq 0$ and $\sin c \neq 0$),

$$\frac{u^2}{\cos^2 c} - \frac{v^2}{\sin^2 c} = 1. \tag{3.44}$$

This is a hyperbola with semiaxes $|\cos c|$ and $|\sin c|$ and foci at ± 1.

We must not think, however, that L coincides with the entire hyperbola. From the equation (3.43) for the hyperbola (parametric

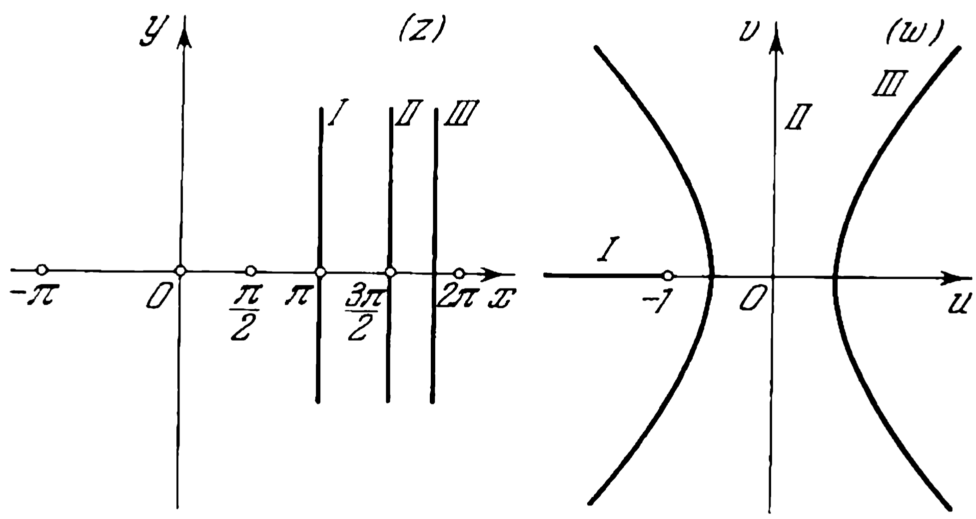

Fig. 23

form) it follows that u retains its sign, which is the sign of $\cos c$, whereas v changes monotonically from $-\infty$ to $+\infty$ (and back). This implies that L coincides with one of the two hyperbola branches (3.43), namely, with the right-hand branch if $\cos c > 0$ and with the left-hand branch if $\cos c < 0$. To the right in Fig. 23 we depict the images of three straight lines in the z-plane:

$$I\,(x=\pi),\quad II\left(x=\frac{3\pi}{2}\right),\ \text{and}\ III\left(x=c,\ \text{where}\ \frac{3\pi}{2}<c<2\pi\right).$$

The mapping of the straight line $z = c + it$ into the corresponding hyperbola branch is one-to-one, and each of the two half-lines into which the straight line is divided by the real axis is mapped in a one-to-one manner into one of the half-branches into which a hyperbola branch is divided in a vertex.

Suppose now that z traverses the straight line l': $z = t + ic'$, parallel to the real axis. The image curve is L':

$$w = \cos z = \cos t \cosh c' - i \sin t \sinh c'.$$

At $c' = 0$ we find that l' coincides with the real axis and L' is given by the equation $w = \cos t \ (-\infty < t < +\infty)$. Hence, w traverses the segment $-1 \leqslant u \leqslant +1$ of the real axis an infinite

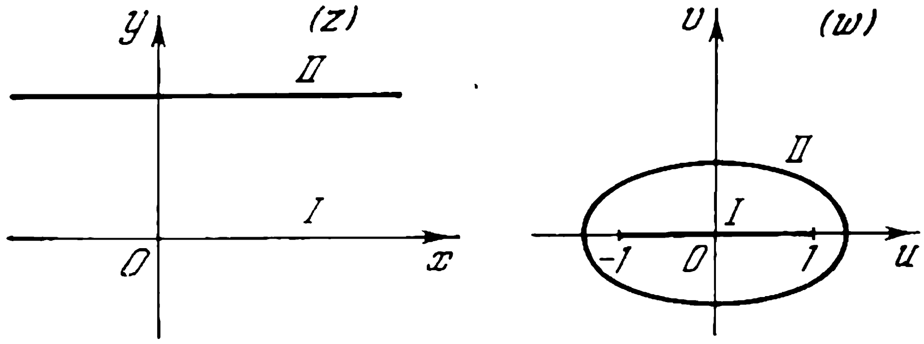

Fig. 24

number of times and to each section of length 2π of the line l' there corresponds a double traverse of the segment. Let $c' \neq 0$. Then we rewrite the equation for curve L' as

$$u = \cos t \cosh c', \ v = -\sin t \sinh c' \ (-\infty < t < \infty) \tag{3.45}$$

and, solving (3.45) for t ($\cosh c' \neq 0$ and $\sinh c' \neq 0$), we arrive at

$$\frac{u^2}{\cosh^2 c'} + \frac{v^2}{\sinh^2 c'} = 1. \tag{3.46}$$

This is an ellipse with semiaxes $|\cosh c'|$ and $|\sinh c'|$ and foci at points ± 1. From the parametric form (3.45) of L' it follows that point w traverses the ellipse in the same direction an infinite number of times and each time it does so point z traverses the straight line $z = t + ic'$ over a distance of 2π (to the right in Fig. 24 we depict the images of two straight lines in the z-plane: I $(y = 0)$ and II $(y = c \neq 0)$).

Therefore, *the mapping* $w = \cos z$ *carries the orthogonal network of straight lines parallel to the coordinate axes into the network of ellipses and hyperbolas with common foci at* ± 1. Since the mapping is conformal in the entire z-plane except at points where $z = k\pi$ ($k=0$, ± 1, ± 2, . . .) (the images of these points are the foci), the network of confocal ellipses and hyperbolas is orthogonal, too. (In Sec. 3.10 we came to the same conclusion.)

3.15

THE GEOMETRIC BEHAVIOR OF TRIGONOMETRIC FUNCTIONS II

Let us take in the z-plane a domain g that by means of function $w = \cos z$ is mapped in a one-to-one manner into a certain domain in the w-plane. There are many ways in which domain g can be chosen, but we must make sure that it does not include two preimages of a point w. Suppose that we have taken a half-strip of width h $(0 < h \leqslant 2\pi)$ that is parallel to the imaginary axis and whose base is

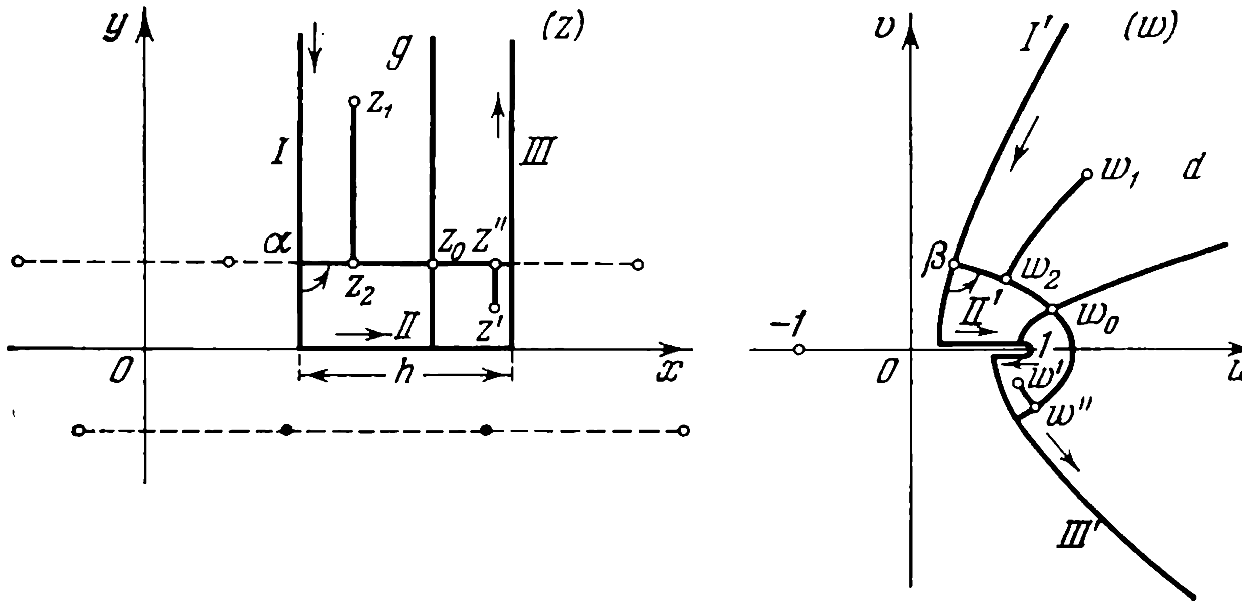

Fig. 25

on the real axis (Fig. 25). It obviously satisfies the abovestated criterion. Indeed, if for a point $z_0 \in g$ we find that $\cos z_0 = w_0$, then all the preimages of w_0 in the z-plane must be situated, as we know (see p. 101), on a straight line passing through z_0 parallel to the real axis (one-half of all the preimages of w_0) and on a straight line symmetric to the first line with respect to the real axis (the other half). But the preimages on the first line are separated from z_0 by distances that are integral multiples of 2π. Since the width of the half-strip does not exceed 2π, not one of the preimages except z_0 get inside the half-strip or on its boundary. The second straight line has no common points with the strip. Hence, the function $w = \cos z$ maps domain g in a one-to-one manner and conformally into a domain in the w-plane.

To build the domain in the w-plane, let us make point z traverse the boundary γ of g in a way such that first it goes along side I of the half-strip, then base II, and finally side III. The point $w = \cos z$ will first describe, continuously and successively, the half-branch I' of one hyperbola, then go along part II' of the curve that

is the image of the section of the real axis $-1 \leqslant u \leqslant 1$, $v = 0$ (since the base of the half-strip is no longer than 2π, point w traverses II' no more than twice), and, finally, describe the half-branch III' of a hyperbola.

The obtained image in the w-plane of the boundary of g (we denote this image by Γ) divides the plane into two parts, one of which is the image d of g (see Sec. 2.4). We give two methods of determining which of the two is actually the image.

The first method, used in Sec. 3.5, consists in taking a point $z_0 \in g$ and marking its image $w_0 = \cos z_0$ in the w-plane. The image cannot belong to Γ, because otherwise one of the preimages of w_0

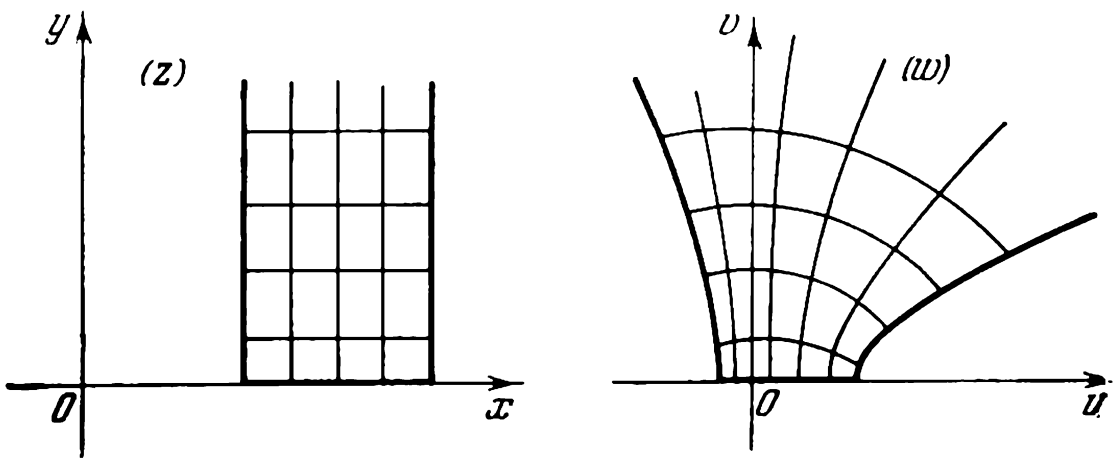

Fig. 26

would belong to g and another to γ, which we saw is impossible. Therefore point w_0 is in one of the two domains in the w-plane, and that domain is the one we are looking for.

The other method, used in Sec. 3.7, consists in fixing the sense of traverse of boundary γ of g. This can be done if, for example, we imagine an observer moving along γ together with point z and noting where the interior of g is. For the sense of traverse indicated in the left diagram in Fig. 25 the domain g is always to the left of the observer. Now we make the observer move along the boundary Γ together with point $w = \cos z$. The image of g should then be to the left of the observer.

Finally, we note that the shape of d will change with the position and width of the half-strip g. Figure 26 depicts a case where the base of the half-strip belongs to a section of the real axis of the type $(k\pi, (k+1)\pi)$. The case depicted in Fig. 25 is peculiar because the base of the half-strip breaks in two, so to say, at $w = 1$ and $w = -1$ as a result of mapping. This happens when the base contains a point of type $k\pi$.

The reader is advised to study the mapping of the strip $0 < x < \pi$ by means of the function $w = \cos z$ by looking at it as the successive

mappings

$$z_1 = iz, \ z_2 = e^{z_1}, \ w = z_3 = \frac{1}{2}\left(z_2 + \frac{1}{z_2}\right).$$

As a result he will find that $w = \cos z$ maps the strip in a one-to-one manner and conformally into a domain in the w-plane bounded by an infinite segment of the real axis that starts at -1, goes through the point at infinity, and ends at 1.

3.16

BRANCHES OF MULTIPLE-VALUED FUNCTIONS

The functions studied in the previous sections assume, generally speaking, one value of w for several (two or more) values of z. An exception is the linear-fractional function, which maps the extended complex plane into itself in a one-to-one manner and conformally. In all the other cases the inverse mapping $z = f^{-1}(w)$ was not unique. This means that the corresponding inverse functions are multivalued (or multiple-valued).

To be able to apply the ideas and results obtained for single-valued functions to multiple-valued functions we must learn how to separate the different branches of such functions. In what follows we will show how this is done.

Let $z = f(w)$ be defined, single-valued, and continuous (in the extended sense) in a domain G of the extended complex plane. We assume that G can be separated in a definite manner into a finite or denumerable set of domains $g_1, g_2, \ldots$ that are pairwise disjoint so that every point of G is an interior point of one and only one domain g_k or a common boundary point for at least two domains g_j and g_k, and at least in one of these domains the mapping $z = f(w)$ is one-to-one. Then, as we already know from Sec. 3.4, the image of each g_k is a domain, $f(g_k) = G_k$, and the entire image $f(G)$ will be covered by the images G_k and also by the images of the common boundaries of the domains g_k.

Let us consider the inverse function $w = F(z)$ in each of the domains G_k. We define it by the additional condition that its values belong to g_k, the preimage of G_k. Then the function $F(z)$, generally multiple-valued, is represented by a collection of a finite or even infinite number of single-valued and continuous (in the extended sense) functions $F_k(z)$. We call each of these a *branch* of the function $F(z)$ in the corresponding domain G_k. It is important to bear in mind that the nature of the G_k together with the corresponding branch $F_k(z)$ depends to a great extent on the way in which we separate G into the g_k. In simple cases G can be separated into such g_k that the corresponding G_k coincide. Assume, for instance, that $G_{k_1}, G_{k_2}, \ldots$ coincide with a certain domain G'. Then the multiple-valued func-

tion $w = F(z)$ has a finite or even infinite number of branches in G', namely $F_{k_1}(z)$, $F_{k_2}(z)$, ...

To the aforesaid we must add that for an arbitrary continuous function $z = f(w)$ a separation of G into the g_k such that the above criteria are met is, generally speaking, impossible. But when $f(w)$ is analytic in G (except at isolated points where it may turn into ∞) such a separation is possible and can be achieved in an infinite number of ways.

A function $z = f(w)$ analytic in a domain g (with the exception, perhaps, of points at which the function becomes ∞) and attaining at different points in G different values ($f(w_1) \neq f(w_2)$ if $w_1 \neq w_2$ and $w_1, w_2 \in g$) is said to be *univalent* (or *simple*) *in domain g*. But if in g there is at least one pair of points at which $f(w)$ attains the same value ($f(w_1) = f(w_2)$ at $w_1 \neq w_2$), the function is said to be *multivalent* in g.

A fact referred to earlier without proof can now be formulated in the following way: if an analytic function $z = f(w)$ is multivalent in a domain G, this domain can be separated into a finite or denumerable set of domains in each of which $f(w)$ is univalent. The corresponding domains g_k are said to be the *univalence domains* of $f(w)$.

Therefore, for functions that are inverses of multivalent functions we can always apply the method of separating of branches discussed earlier.

We will illustrate this method for elementary functions; the separation of G into univalence domains will be achieved by using the known properties of the elementary functions.

Apart from functions that are inverses of elementary functions we consider also other many-valued functions, which are composite functions of the type $\varphi(z)\,\psi(z)$ (where $\varphi(z)$ or $\psi(z)$ is the inverse of an elementary function) or rational combinations of such functions.

3.17

THE FUNCTION $w = \sqrt[n]{z}$

This is the inverse of $z = w^n$ (where n is a positive integer greater than unity).

For each value of z different from 0 and ∞, the function has n different values given by the formula

$$w = \sqrt[n]{|z|}_{+}\left(\cos\frac{\operatorname{Arg} z}{n} + i\sin\frac{\operatorname{Arg} z}{n}\right). \tag{3.47}$$

At $z = 0$ and $z = \infty$ we obtain only one value: $w = 0$ and $w = \infty$, respectively.

The n values given by (3.47), which represent the points in the w-plane at which w^n attains the same value z, are situated at the

vertices of a regular n-gon inscribed in the circle $| w | = \sqrt[n]{| z |}$ (+).

Conversely: the vertices of a regular n-gon with its center at the origin of coordinates can be considered as the n values of $\sqrt[n]{z}$. For this reason a domain g in the w-plane is the univalence domain of $z = w^n$ if and only if it contains not more than one vertex out of the totality of n vertices of the regular n-gon with its center at $w = 0$. Obviously, this condition is met by any angle with its vertex at the origin of coordinates and an opening span of $2\pi/n$.

From the origin of coordinates let us draw n rectilinear rays at equal angles. We can then see that the entire plane in which the multivalent function $z = w^n$ is defined is separated into n univalence domains of this function: $g_1, g_2, \ldots, g_n$. The image of each of these domains is the same domain G' in the z-plane whose boundary is a rectilinear ray L starting at the origin of coordinates. If domain g_k is bounded by rays that make angles $\varphi_0 + 2k\pi/n$ and $\varphi_0 + 2(k + 1)\pi/n$ with the positive direction of the real axis, respectively, then ray L makes an angle $n\varphi_0$ with the positive direction of the real axis.

According to the results of Sec. 3.16, in G' we have n branches of the function $\sqrt[n]{z}$. Each of these branches, $\underset{k}{\sqrt[n]{z}}$ $(k = 1, 2, \ldots, n)$, is fully defined by the condition that it belong to the domain g_k. Since the function $z = w^n$ has a nonzero derivative at all points in g_k $(z' = nw^{n-1})$, the $\underset{k}{\sqrt[n]{z}}$ have nonzero derivatives:

$$\left(\underset{k}{\sqrt[n]{z}}\right)' = \frac{1}{nw^{n-1}} = \frac{1}{n\left(\underset{k}{\sqrt[n]{z}}\right)^{n-1}}.$$

Let us now take a system of rectilinear rays emerging from the origin of coordinates and that is obtained from the previous one by rotation about the origin of coordinates through an angle α $(0 < \alpha < 2\pi/n)$. Then the new system separates the w-plane into n domains $d_1, d_2, \ldots, d_n$ each of which consists of parts of two "old" domains, say d_k consists partly of g_k and partly of g_{k+1} (if $k = n$, then we must substitute g_1 for g_{n+1}) (Fig. 27).

The image of each of the domains d_k in the w-plane is the same domain D' bounded by the rectilinear ray M emerging from the origin of coordinates at an angle $n\varphi_0 + n\alpha$ to the positive direction of the real axis. In this domain we also have n branches of $\sqrt[n]{z}$, each of which is fully defined by the condition that it belong to the corresponding domain, say d_k. We denote these branches by $(\sqrt[n]{z})_k$. They are differentiable in D':

$$(\sqrt[n]{z})'_k = 1/[n(\sqrt[n]{z})_k^{n-1}].$$

Let us compare them with the branches $\underset{k}{\sqrt[n]{z}}$. Since part of the preimage d_k of D' in the w-plane belongs to g_k and the other part to g_{k+1}, the branch $(\sqrt[n]{z})_k$ in a part of D' (the image of the common part of d_k and g_k) coincides with $\underset{k}{\sqrt[n]{z}}$ and in the other part of D' (the image of the common part of d_k and g_{k+1}) coincides with $\underset{k+1}{\sqrt[n]{z}}$.

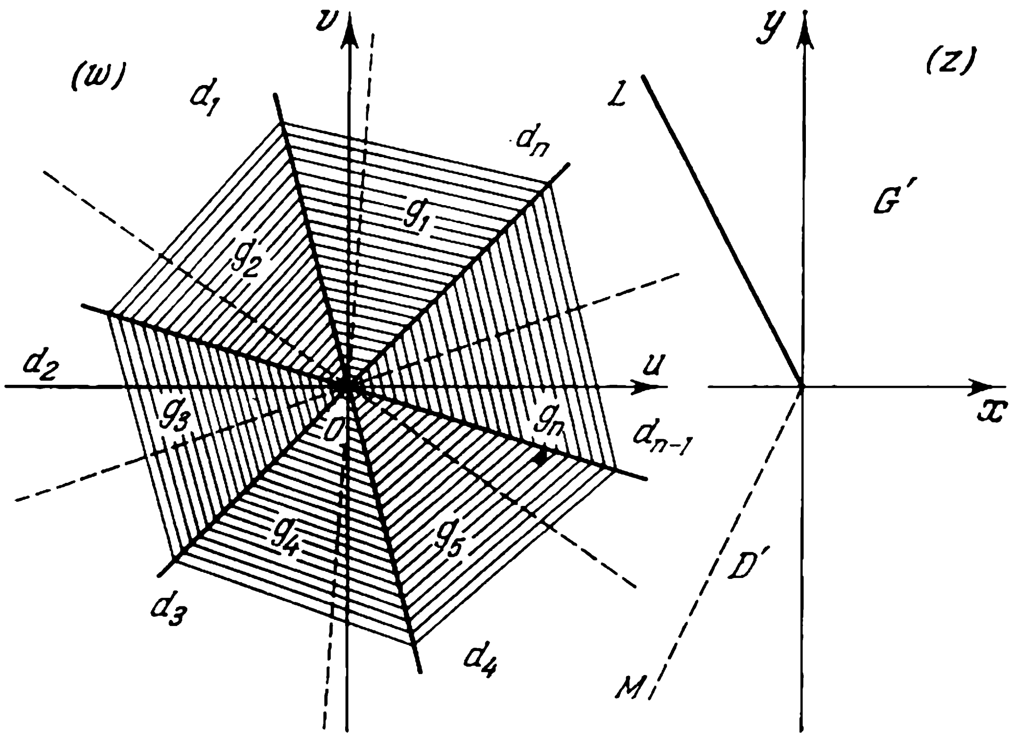

Fig. 27

We see that when one set of univalence domains is substituted for another, a new branch is obtained by linking one old branch with another old branch.

If the rotation angle $\alpha = 0$, then d_k coincides with g_k, domain D' with G', and branch $(\sqrt[n]{z})_k$ with $\underset{k}{\sqrt[n]{z}}$. But if α approaches $2\pi/n$ by continuously growing, then d_k approaches g_{k+1}, the corresponding domain D' approaches G', and branch $(\sqrt[n]{z})_k$ over a larger and larger part of D' coincides with $\underset{k+1}{\sqrt[n]{z}}$ (instead of g_{n+1} and $\underset{n+1}{\sqrt[n]{z}}$ we must take g_1 and $\underset{1}{\sqrt[n]{z}}$). At $\alpha = 2\pi/n$, d_k coincides with g_{k+1}, D' with G', and $(\sqrt[n]{z})_k$ with $\underset{k+1}{\sqrt[n]{z}}$.

We can follow the transition from one branch $\underset{k}{\sqrt[n]{z}}$ to another branch $\underset{k+1}{\sqrt[n]{z}}$ by making point z traverse a circle with its center at the origin of coordinates. If at point z_0 the value of $\sqrt[n]{z}$ is

taken on the branch $\sqrt[n]{z}_{k}$ and corresponds to point w_0 in g_k, i.e.

$$w_0 = \sqrt[n]{|z_0|}_{+} \left(\cos\frac{\psi_0}{n} + i\sin\frac{\psi_0}{n}\right).$$

then as the point z moves along the circle $|z| = |z_0|$ in the positive sense, the corresponding value of the function

$$w = \sqrt[n]{|z_0|}_{+} \left(\cos\frac{\psi}{n} + i\sin\frac{\psi}{n}\right),$$

continuously changes together with ψ, and after z makes a full circle and again becomes z_0, the value of the function becomes

$$w_1 = \sqrt[n]{|z_0|}_{+} \left(\cos\frac{\psi_0+2\pi}{n} + i\sin\frac{\psi_0+2\pi}{n}\right).$$

This value is obtained from w_0 by rotation about the origin of coordinates through an angle of $2\pi/n$; hence, point w_1 belongs to g_{k+1} (which is adjacent to g_k) and is a value of the function on branch $\sqrt[n]{z}_{k+1}$ at point z_0.

This conclusion remains valid for any point in G', which implies that as a result of point z traversing a circle of any radius with its center at the origin of coordinates the values of $\sqrt[n]{z}$ continuously change and go from branch $\sqrt[n]{z}_{k}$ to branch $\sqrt[n]{z}_{k+1}$.

For the branches of $\sqrt[n]{z}$ to pass from one to another and return to the initial one (from $\sqrt[n]{z}_{k}$ to $\sqrt[n]{z}_{k+1}$, from $\sqrt[n]{z}_{k+1}$ to $\sqrt[n]{z}_{k+2}$, ..., from $\sqrt[n]{z}_{n}$ to $\sqrt[n]{z}_{1}$, ..., and from $\sqrt[n]{z}_{k-1}$ to $\sqrt[n]{z}_{k}$), point z must circle point $z=0$ in the positive sense n times.

A point possessing the property that (one) full circle around it in a sufficiently small neighborhood along a Jordan curve changes one branch of a multiple-valued function to another branch of the same function is called a *branch point* of the function. The fact that after a point has been circled n times in the same direction we arrive at the initial branch is stated differently by saying that the branch point has a *degree of ramification* equal to $n - 1$, or a *multiplicity* equal to n; the branch point is then called *algebraic*.*

Hence, point $z = 0$ is an algebraic branch point of multiplicity n or degree of ramification $n - 1$ of the function $\sqrt[n]{z}$.

Obviously, we can consider point $z = \infty$ another algebraic branch point of $\sqrt[n]{z}$ of multiplicity n because a traversal of a circle of arbi-

* The last notion assumes also that the function has a limit (finite or infinite) at this point.

trarily large radius with its center at the origin of coordinates is actually a traversal about point $z = 0$. Thus, the function $\sqrt[n]{z}$ has two algebraic branch points in the z-plane, $z = 0$ and $z = \infty$, both of multiplicity n or degree of ramification $n - 1$.

The branches of the function $\sqrt[n]{z}$ described above were built for domains G' and D', whose boundary was a ray connecting the two branch points. We will arrive at a more general type of domain if instead of a straight ray we take an arbitrary Jordan arc in the extended plane, i.e. a curve connecting 0 and ∞. We denote this curve by Γ and the domain whose boundary it is by G. If z traverses Γ from the initial point (0) to the finite point (∞), the corresponding n points of $w = \sqrt[n]{z}$ traverse n Jordan arcs γ_k ($k = 1, 2, \ldots, n$) connecting points 0 and ∞. These n curves do not have any common points except 0 and ∞ and constitute (pairwise) Jordan curves in the extended complex plane.

Suppose that g'_k is one of the two domains bounded by a pair of curves γ_k and γ_{k+1} that does not contain the curves $\gamma_1, \ldots, \gamma_{k-1}, \gamma_{k+1}, \ldots, \gamma_n$. If we rotate the w-plane about the origin of coordinates through an angle of $2\pi/n$, then, in view of the construction, γ_k goes over to γ_{k+1} and γ_{k+1} goes over to γ_{k+2}, and domain g'_k goes over to g'_{k+1}. Since g'_k has no common points with g'_{k+1}, not one of these two domains has points that turn into each other as a result of the rotation. Therefore, all the g'_k are univalence domains for $z = w^n$, and we obtain n branches of $\sqrt[n]{z}$ in G if we require that the values of each branch belong to the corresponding domain g'_k. To fix one branch it suffices to indicate a value of $\sqrt[n]{z}$ at a point z_0 in G. If this value is w_0, there exists a unique domain g'_k with point w_0 and a unique branch of $\sqrt[n]{z}$ in G that admits the value w_0 at z_0. This is the customary method of fixing a definite branch of $\sqrt[n]{z}$ in G.

Let $\underset{k}{\sqrt[n]{z}}$ and $\underset{l}{\sqrt[n]{z}}$ be two branches of $\sqrt[n]{z}$ in G and let w'_0 and w''_0 be two values on these branches at a point z_0. Since

$$w'_0 = \underset{k}{\sqrt[n]{z}} = \underset{+}{\sqrt[n]{|z_0|}} \left(\cos \frac{\varphi_0 + 2m'\pi}{n} + i \sin \frac{\varphi_0 + 2m'\pi}{n}\right),$$

$$w''_0 = \underset{l}{\sqrt[n]{z}} = \underset{+}{\sqrt[n]{|z_0|}} \left(\cos \frac{\varphi_0 + 2m''\pi}{n} + i \sin \frac{\varphi_0 + 2m''\pi}{n}\right),$$

where m' and m'' are integers, we can obtain w''_0 from w'_0 by multiplying the latter by

$$\eta = \cos \frac{2(m'' - m')\pi}{n} + i \sin \frac{2(m'' - m')\pi}{n},$$

i.e. by one of the values of $\sqrt[n]{1}$. But by multiplying $\underset{k}{\sqrt[n]{z}}$ by η we, obviously, arrive at a single-valued and continuous in G function

$\eta_k \sqrt[n]{z}$ whose values represent $\sqrt[n]{z}$ and belong to the same domain as the point $\eta_k \sqrt[n]{z_0} = \sqrt[n]{z_0}_l$. Hence, $\eta_k \sqrt[n]{z} = \sqrt[n]{z}_l$ everywhere in G. We see that two branches of $\sqrt[n]{z}$ in the same domain G can be obtained by multiplying one of the two by a value of $\sqrt[n]{1}$.

All statements made in this section can be carried over, with appropriate alterations, to functions of a somewhat more general form,

$$w = \sqrt[n]{z-a} \quad \text{or} \quad w = \sqrt[n]{\frac{z-a}{z-b}}.$$

The reader is advised to examine these cases. (Note that these functions are inverses of $z = a + w^n$ and $z = (bw^n - a)/(w^n - 1)$, which have the same univalence domains as the function $z = w^n$.) He will find that the branch points of $w = \sqrt[n]{z-a}$ are a and ∞, the branch points of $w = \sqrt[n]{(z-a)/(z-b)}$ are a and b, and it is possible to isolate a branch of each function in any domain whose boundary is a Jordan arc connecting the branch points.

3.18

THE FUNCTION $w = \sqrt[n]{P(z)}$

To clarify the notion of a branch point we consider the multiple-valued function

$$w = f(z) = \sqrt[n]{P(z)}, \tag{3.48}$$

where $P(z)$ is an arbitrary polynomial. Suppose that N is the degree of this polynomial, $a_1, a_2, \ldots, a_m$ its various zeros, and $\alpha_1, \alpha_2, \ldots, \alpha_m$ the multiplicities of the zeros ($\alpha_1 + \alpha_2 + \ldots + \alpha_m = N$). Then we can write

$$P(z) = A(z - a_1)^{\alpha_1} \ldots (z - a_m)^{\alpha_m},$$

whence

$$f(z) = \sqrt[n]{A(z-a_1)^{\alpha_1} \ldots (z-a_m)^{\alpha_m}}. \tag{3.49}$$

Let us consider an arbitrary Jordan curve γ (a circle, for instance) that does not pass through a single point a_k ($k = 1, \ldots, m$). We make point z traverse this curve once in a definite direction, and also fix the arguments of $z - a_1, \ldots, z - a_m$ at some point z_0 on γ. Let us assume that these arguments are $\varphi_1^{(0)}, \ldots, \varphi_m^{(0)}$. As point z traverses γ, the angle φ_k between vector $z - a_k$ and the positive direction of the real axis continuously changes starting from the initial value $\varphi_k^{(0)}$. As a result of a single traversal of γ the angle either returns to its previous value $\varphi_k^{(0)}$ (if point a_k lies in the exterior of

γ) or increases by $\pm 2\pi$ (if point a_k lies in the interior of γ) (Fig. 28).* The sign $+$ or $-$ depends on the choice of direction of traversal of γ; we call positive the direction in which the angles receive a positive increment 2π. For the sake of definiteness we assume point z traverses γ in the positive sense. If not one of the points a_k lies inside γ, all angles φ_k return as a result of traversal to their initial values $\varphi_k^{(0)}$, and so does the function (3.49). This implies that there is not a single finite point ζ in the plane that differs from the a_k but is still

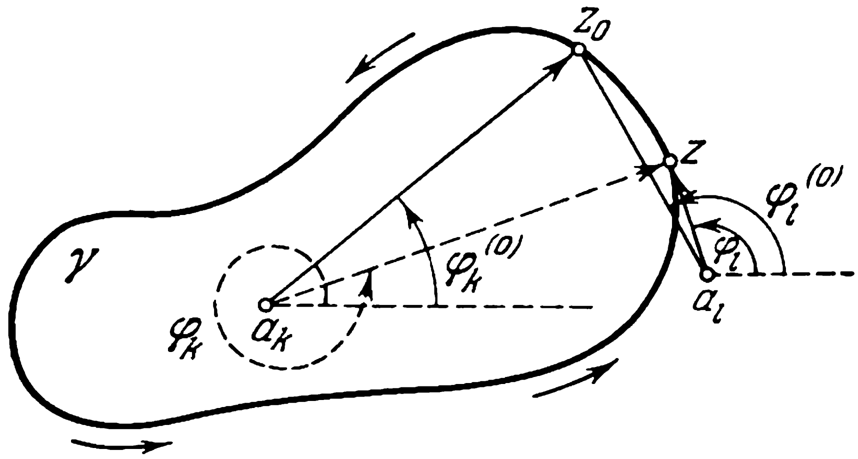

Fig. 28

a branch point for $f(z)$. True, since if for such a point ζ we choose a neighborhood that does not contain any of the a_k, the traversal of any Jordan curve in this neighborhood containing ζ as its interior point conserves the chosen branch of the function.

Hence, *there is not a single point ζ different from the a_k that is a branch point of $f(z)$.*

Now let us consider a neighborhood of a point a_k so small that the other branch points, $a_1, \ldots, a_{k-1}, a_{k+1}, \ldots, a_m$, are excluded. Then in the traversal of Jordan curve γ that belongs to this neighborhood and contains a_k as its interior point, the angle φ_k will grow by 2π while the other angles, $\varphi_1, \ldots, \varphi_{k-1}, \varphi_{k+1}, \ldots, \varphi_m$, return to their initial values. This implies that the argument of the radicand in (3.49) changes by $2\pi\alpha_k$ as a result of the traversal; whence the function (3.49) acquires a factor $\cos\frac{2\pi\alpha_k}{n} + i\sin\frac{2\pi\alpha_k}{n}$, which differs from unity if and only if α_k is not divisible by n. Therefore, each zero point a_k of $P(z)$ whose α_k is not divisible by n is a branch point of $\sqrt[n]{P(z)}$. To determine the degree of ramification of this branch point, we assume that δ_k ($\delta_k < n$) is

* These facts are easily verified in the simple cases (e.g. when γ is a circle, ellipse, or polygon) but can be rigorously proved in the general case. One proof is given in P. S. Aleksandrov, *Combinatorial Topology* [in Russian], Gostekhizdat, Moscow, 1947, Chap. 2.

the greatest common divisor of α_k and n. Then if we put $\alpha_k = \delta_k \alpha_k'$ and $n = \delta_k \nu_k$ ($\nu_k > 1$), we can write $\cos \frac{2\pi\alpha_k}{n} + i \sin \frac{2\pi\alpha_k}{n}$ in the form $\cos \frac{2\pi\alpha_k'}{\nu_k} + i \sin \frac{2\pi\alpha_k'}{\nu_k}$.

As a result of traversal of curve γ p times in the same direction the function $f(z)$ acquires a factor $\cos \frac{2\pi\alpha_k' p}{\nu_k} + i \sin \frac{2\pi\alpha_k' p}{\nu_k}$, which is equal to unity if and only if p is divisible by ν_k. The smallest possible value of p then is ν_k, which implies that the degree of ramification of the branch point is $\nu_k - 1$.

Finally, let us consider a neighborhood of the point at infinity not containing any of the branch points a_k and in this neighborhood a Jordan curve γ containing in its interior all the a_k. Then the exterior of γ contains the point ∞ and not a single point a_k. We traverse γ completely one time. All angles φ take on an increment of 2π and, hence, the argument of the radicand in (3.49) changes by $2\pi(\alpha_1 + \alpha_2 + \ldots + \alpha_m)$. The function $f(z)$ then acquires the factor

$$\cos \frac{2\pi(\alpha_1 + \ldots + \alpha_m)}{n} + i \sin \frac{2\pi(\alpha_1 + \ldots + \alpha_m)}{n} = \cos \frac{2\pi N}{n} + i \sin \frac{2\pi N}{n}.$$

It is either unity or not unity depending on whether N is divisible by n. In the first case ∞ is not a branch point and in the second it is. Moreover, if δ is the greatest common divisor of N and n ($\delta < N$) and if $n = \delta\nu$, the degree of ramification of the point at infinity considered as a branch point is $\nu - 1$.

Thus, when α_k is divisible by n, the traversal of the Jordan curve γ containing in its interior point a_k and not a single remaining branch point a_j does not change the value of $f(z)$. In the same way a complete traversal of a Jordan curve γ containing all the a_k does not change the values of $f(z)$ if N is divisible by n.

In general, let $a_{k_1}, \ldots, a_{k_q}$ be a group of branch points for which the sum $\alpha_{k_1} + \ldots + \alpha_{k_q}$ is divisible by n. Then the complete traversal of any Jordan curve γ containing in its interior the above-mentioned branch points and not containing a single branch point out of the group cannot change the values of $f(z)$. For this reason in any domain G containing only Jordan curves whose interiors either contain no branch point a_k or contain groups of branch points for which the sums of the corresponding α_k's are divisible by n, we can isolate branches of the function $f(z)$.

To this end it suffices to fix the value w_0 of $f(z)$ at one of the points z_0 in this domain. Among the n images $f(G)$ of G in the w-plane one contains point w_0; we denote this image domain by g_k. Then we can completely define a branch of $f(z)$ in G by the condition that all its values belong to g_k. The value on this branch at a point z_1

in G can be obtained in the following way. Connect point z_0 with z_1 by a continuous curve γ_1 belonging to G and traverse it from z_0 to z_1 watching that the corresponding value of $f(z)$ continuously changes starting at w_0. Then we arrive at point z_1 with one of the n values of $f(z)$, which we denote by w_1. This value depends only on the value w_0 chosen at point z_0 and on the point z_1 but does not depend on the choice of the path from z_0 to z_1 and, hence, is a single-valued function of z_1 in G. Indeed, if γ_2 is another curve connecting z_0 and z_1 in G, then, as we traverse the entire closed curve γ consisting of

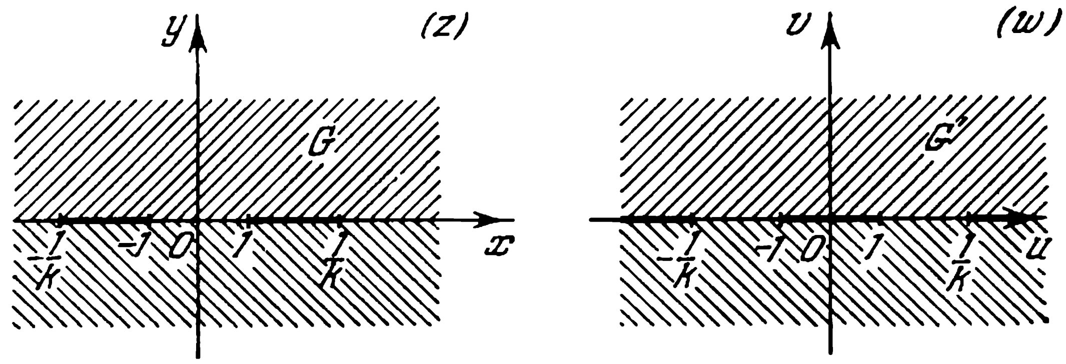

Fig. 29

γ_1 and γ_2, we arrive when traveling from z_0 to z_1 along γ_1 at value w_1 at point z_1 and then, returning to z_0 along γ_2, we must arrive at the initial value w_0 (since we already know that the traversal of a closed curve in G cannot change the value of $f(z)$). From this it follows that moving along γ_2 from z_0 to z_1 we arrive at point z_1 at the same value w_1 as we did when moving along γ_1.

Here are three examples to clarify the aforesaid.

(1) $w = \sqrt{(1-z^2)(1-k^2z^2)}$, where $0 < k < 1$. This is a two-valued function with four branch points: ± 1 and $\pm 1/k$. Here $N = 4$ is divisible by $n = 2$ and, therefore, ∞ is not a branch point.

Since all the α_k are equal to unity (± 1 and $\pm 1/k$ are simple zeros of the radicand), a complete traversal of any closed curve γ containing in its interior only two branch points does not change the value of w. For this reason we can isolate branches of w in the following domains (for example): a domain G whose boundary consists of two segments, $-1/k \leqslant x < -1$ and $1 \leqslant x \leqslant 1/k$, or a domain G' whose boundary consists of the segment $-1 \leqslant x \leqslant 1$ and the infinite segment of the real axis starting at $-1/k$, passing through ∞, and ending at $1/k$ (Fig. 29).

In the first domain the branches $f_1(z)$ and $f_2(z)$ can be distinguished by the values they assume at the origin of coordinates $f_1(0) = 1$ but $f_2(0) = -1$.

(2) $w = \sqrt{4z^3 - g_2 z - g_3}$, where g_2 and g_3 are complex numbers for which $g_2^3 - 27g_3^2 \neq 0$ (this means that the discriminant of the polynomial $4z^3 - g_2z - g_3$ is nonzero and, hence, the zeros of the

polynomial, e_1, e_2, e_3, are different). Since here $N = 3$ is not divisible by $n = 2$, point ∞ is also a branch point. Complete traversal of a Jordan curve containing in its interior a pair of branch points does not change the value of the function (just as in the previous example). Therefore, connecting point e_1 with e_2, and point e_3 with ∞ by Jordan arcs γ_1 and γ_2, respectively, we arrive at a domain G with a boundary consisting of γ_1 and γ_2; in this domain we can isolate the branches of w (Fig. 30).

(3) Let us consider a function that is the inverse of $z = \frac{1}{2}(w + 1/w)$, i.e. $w = \varphi(z) = z + \sqrt{z^2 - 1}$. This is a two-valued function with the same branch points as $\sqrt{z^2 - 1}$, i.e. ± 1.

To find a domain G where we can isolate the branches of this function, we connect points -1 and 1 by a segment of the real axis. We

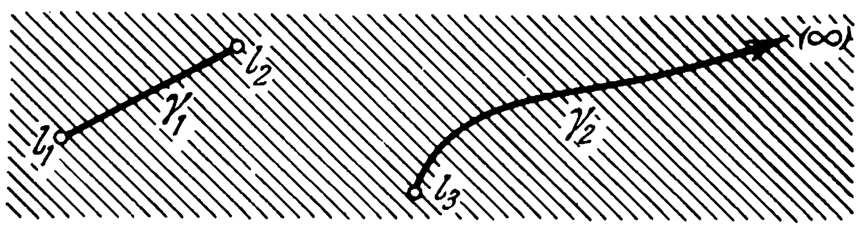

Fig. 30

arrive at a domain which is mapped by the function $w = z + \sqrt{z^2 - 1}$ in a one-to-one manner into two domains, the interior of the unit circle and its exterior (g_1 and g_2; see Sec. 3.10). Either can be isolated by fixing one of the two values of w at a point in G (e.g. at the point at infinity). From the formula $z = \frac{1}{2}(w + 1/w)$ we can see that z turns into ∞ either at $w = 0$ or at $w = \infty$. Therefore, one of the branches of $\varphi(z)$ is characterized by the fact that on it $\varphi(\infty) = 0$; this branch maps G into the interior of the unit circle. On the other branch $\varphi(\infty) = \infty$; it maps G into the exterior of the unit circle.

Instead of G we could have taken another domain, G', whose boundary consists of the infinite segment of the real axis connecting the points -1 and 1 (through ∞), or even the domains G'' and G''' with boundaries that are the upper and lower semicircles, respectively.

Using the results obtained in Sec. 2.10, the reader can examine the domains in the w-plane into which the branches of $\varphi(z)$ map the domains G', G'', and G'''.

The contents of this section concern a multiple-valued function of the type $\sqrt[n]{P(z)}$, where $P(z)$ is a polynomial, but the reader can extend these results without difficulty to the more general case of

a function of the type $\sqrt[n]{R(z)}$, where $R(z)$ is an arbitrary rational function. To find the branch points of $\sqrt[n]{R(z)}$ one must examine all the points in the complex plane where $R(z)$ turns into 0 or ∞.

3.19

THE LOGARITHM

The function that is the inverse of

$$z = e^w = e^u (\cos v + i \sin v)$$

is defined for every z except 0 and ∞ and is written in the form (see (3.24))

$$w = \ln |z| + i \operatorname{Arg} z.$$

The function is obviously multiple-valued and even infinitely valued; it is called the *logarithm of* z and is denoted by $\operatorname{Ln} z$. Hence, by definition,

$$w = \operatorname{Ln} z = \ln |z| + i \operatorname{Arg} z. \tag{3.50}$$

If we call the value of $\operatorname{Ln} z$ equal to $\ln |z| + i \arg z$ the *principal value* and denote it by $\ln z$, we can write

$$\operatorname{Ln} z = \ln z + 2k\pi i, \tag{3.51}$$

where $k = 0, \pm 1, \pm 2, \ldots$.

It follows that *each complex number differing from zero or infinity has an infinite number of logarithms (i.e. values of the logarithmic function) of which any two differ by a term that is an integral multiple of* $2\pi i$. If z is a real positive number, the principal value of the logarithm coincides with $\ln |z|$ and, hence, is a real number. This number is known from the general calculus course as the (real-valued) logarithm of a (real) positive number. For instance, $\ln 1 = 0$, $\ln e = 1$, and $\ln 2 = 0.693\,147\,18\ldots$.

But apart from real values, the logarithm of real positive numbers has an infinitude of imaginary values, according to (3.51). For instance, $\operatorname{Ln} 1 = 2k\pi i$, $\operatorname{Ln} e = 1 + 2k\pi i$, and $\operatorname{Ln} 2 = 0.693\,147\,18\ldots + 2k\pi i$.

For real negative numbers and imaginary numbers the principal value of the logarithm is the imaginary number

$$\ln |z| + i \arg z \quad (\arg z \neq 0, \ |\arg z| \leqslant \pi).$$

All other values of the logarithm are imaginary numbers, too, and are calculated by (3.51). For instance,

$$\operatorname{Ln}(-1) = (2k+1)\pi i, \quad \operatorname{Ln}(-2) = 0.693\,147\,18\ldots + (2k+1)\pi i,$$

$$\operatorname{Ln}(1-i) = \ln \sqrt{2} + i\left(-\frac{\pi}{4} + 2\pi k\right)$$
$$= 0.346\,573\,59\ldots + (8k-1)\frac{\pi i}{4}.$$

The well-known rules concerning the logarithm of a product and a ratio remain valid for the (multiple-valued) logarithm of a complex number; namely,

$$\begin{aligned} \mathrm{Ln}\,(z_1 z_2) &= \ln |z_1 z_2| + i \,\mathrm{Arg}\,(z_1 z_2) \\ &= \ln |z_1| + \ln |z_2| + i\,(\mathrm{Arg}\, z_1 + \mathrm{Arg}\, z_2) = \mathrm{Ln}\, z_1 + \mathrm{Ln}\, z_2, \end{aligned} \tag{3.52}$$

$$\begin{aligned} \mathrm{Ln}\,\frac{z_1}{z_2} &= \ln \left|\frac{z_1}{z_2}\right| + i\,\mathrm{Arg}\,\frac{z_1}{z_2} = \ln |z_1| - \ln |z_2| \\ &\quad + i\,(\mathrm{Arg}\, z_1 - \mathrm{Arg}\, z_2) = \mathrm{Ln}\, z_1 - \mathrm{Ln}\, z_2, \end{aligned} \tag{3.53}$$

where z_1 and z_2 are arbitrary but nonzero complex numbers. Both in (3.52) and (3.53) the left- and right-hand sides depict, for given z_1 and z_2, infinite sets of complex numbers. The equalities mean that these sets are identical, i.e. consist of the same numbers. The omission of this fact often leads to mistakes.

Consider the following sophism attributed to Johann Bernoulli. It is stated that $\mathrm{Ln}\,(-z) = \mathrm{Ln}\, z$ for all nonzero values of z. The "proof" is as follows:

(1) $\mathrm{Ln}\,[(-z)^2] = \mathrm{Ln}\,(z^2)$, (2) $\mathrm{Ln}\,(-z) + \mathrm{Ln}\,(-z) = \mathrm{Ln}\, z + \mathrm{Ln}\, z$,

(3) $2\,\mathrm{Ln}\,(-z) = 2\,\mathrm{Ln}\, z$, and (4) $\mathrm{Ln}\,(-z) = \mathrm{Ln}\, z$.

But the statement is wrong because

$$\mathrm{Ln}\, z = \ln |z| + i\,\mathrm{Arg}\, z = \ln |z| + i \arg z + 2k\pi i,$$

$$\mathrm{Ln}\,(-z) = \ln |-z| + i\,\mathrm{Arg}\,(-z) = \ln |z| + i \arg z + (2k+1)\pi i.$$

Obviously, not a single number that is a value of $\mathrm{Ln}\, z$ is equal to the values of $\mathrm{Ln}\,(-z)$.

The mistake lies in the incorrect transition from step (2) to step (3). The first is, of course, correct because it is based on (3.52). But we cannot replace the sum $\mathrm{Ln}\,(-z) + \mathrm{Ln}\,(-z)$ by $2\,\mathrm{Ln}\,(-z)$ since the sum is obtained from the set of numbers $\mathrm{Ln}\,(-z)$ by adding any number from this set and an equal or different number from the same set, whereas the set of numbers $2\,\mathrm{Ln}\,(-z)$ is obtained by adding only equal numbers, i.e. by doubling them. Therefore, $\mathrm{Ln}\,(-z) + \mathrm{Ln}\,(-z) \neq 2\,\mathrm{Ln}\,(-z)$. Similarly,

$$\mathrm{Ln}\, z + \mathrm{Ln}\, z \neq 2\,\mathrm{Ln}\, z.$$

This becomes still more obvious if we consider the following example. Suppose A is a set whose only elements are 0 and 1. Then $A + A$ denotes a set with three elements: $0 + 0 = 0$, $0 + 1 = 1$, and $1 + 1 = 2$, whereas the set $2A$ is a set with only two elements: $2 \times 0 = 0$ and $2 \times 1 = 2$.

Note that if in (3.53) we put $z_1 = z_2 = z \neq 0$,

$$\mathrm{Ln}\, 1 = \mathrm{Ln}\, z - \mathrm{Ln}\, z.$$

This relationship is valid, but the right-hand side cannot be substituted by 0 because we are dealing with a set consisting of differences between values of the logarithm of the same number. The set then consists of numbers that are integral multiples of $2\pi i$, so that we arrive at the identity

$$\mathrm{Ln}\,1 = 2k\pi i \quad (k = 0, \pm 1, \pm 2, \ldots).$$

Going on to a study of the branches of the logarithm, we will start by establishing the univalence domains of the function $z = e^w$, for which the logarithm is the inverse function.

Since all the values of w for which e^w assumes a given value of z ($z \neq 0$ and $z \neq \infty$) are given by the formula (3.24)

$$w = \ln |z| + i \operatorname{Arg} z$$

and can be obtained from any one value by adding the term $2k\pi i$ ($k = \pm 1, \pm 2, \ldots$), the univalence domain of the exponential function must not contain any points one of which is obtained from the other by such a shift.

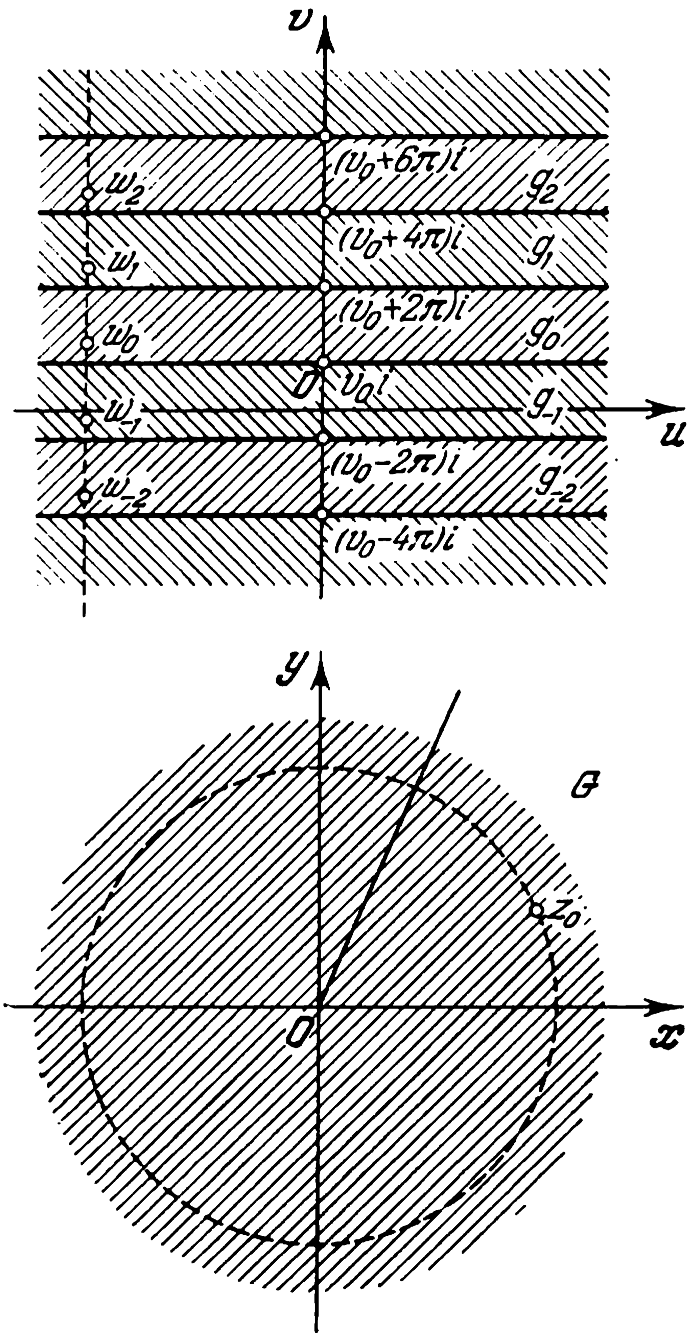

Fig. 31

The simplest way to satisfy these conditions is to take a strip g_0 parallel to the real axis and having a width of 2π, i.e. $v_0 < v < v_0 + 2\pi$. Along with this univalence domain there is an infinite number of univalence domains g_k, all of which are strips parallel to the real axis: $v_0 + 2k\pi < v < v_0 + (2k + 2)\pi$ ($k = \pm 1, \pm 2, \ldots$).

It is obvious that each point of the w-plane belongs either to the interior of one of the univalence domains g_k (which includes g_0, too) or to the boundary between two such domains g_k and g_{k+1} (Fig. 31). The same domain G in the z-plane is the image of each strip g_k ($k = 0, \pm 1, \pm 2, \ldots$); namely, G is an angle with an opening span of 2π and the vertex at the origin of coordinates. The boundary of G is the rectilinear ray starting at the origin of coordinates and having an angle v_0 with the positive direction of the real axis.

In G, therefore, we have an infinite (denumerable) set of the various branches of $\mathrm{Ln}\, z$. Each of these branches $\mathrm{Ln}_k z$ is fully characterized by the fact that its values belong to a definite strip g_k, although it is quite sufficient to fix a value w_0 of $\mathrm{Ln}\, z$ at some point z_0 of G because of all the univalence domains g_k only one strip g_{k_0} contains point w_0.

Consider the branch

$$\mathrm{Ln}_k z = \ln |z| + i \,\mathrm{Arg}_k z,$$

where $\mathrm{Arg}_k z$ is the value of the argument satisfying the condition

$$v_0 + 2k\pi < \mathrm{Arg}_k z < v_0 + (2k+2)\pi.$$

(This condition means that the values of $\mathrm{Ln}_k z$ belong to strip g_k.)

Since the function $w = \mathrm{Ln}_k z$ maps in a one-to-one manner and continuously the domain G into the strip g_k and its inverse is $z = e^w$ (a function that has a nonzero derivative everywhere in g_k), the rule of differentiation of inverse functions implies that $\mathrm{Ln}_k z$ also has a derivative calculated via the formula

$$(\mathrm{Ln}_k z)' = \frac{1}{(e^w)'} = \frac{1}{e^w} = \frac{1}{z}.$$

The branch points of $\mathrm{Ln}\, z$ are zero and ∞. Indeed, when z performs a complete traversal of a circle with its center at the origin of coordinates (a circle of an arbitrarily small or large radius), the value of $\mathrm{Arg}\, z$ varies, starting at an initial value $\mathrm{Arg}_k z_0$, and receives after completion of the circle an increment $\pm 2\pi$ (the sign depends on the sense of traversal); therefore, the branch

$$\mathrm{Ln}_k z = \ln |z| + i \,\mathrm{Arg}_k z$$

transforms into another branch

$$\mathrm{Ln}_{k\pm 1} z = \ln |z| + i (\mathrm{Arg}_k z \pm 2\pi) = \ln |z| + i \,\mathrm{Arg}_{k\pm 1} z.$$

Obviously, if we repeat the process of traversal an infinite number of times in one direction (e.g. positive), after completion of each circle we arrive at the next branch,

$$\mathrm{Ln}_{k+2} z,\ \mathrm{Ln}_{k+3} z,\ \ldots,$$

and, hence, we can never return by this process to the initial branch $\mathrm{Ln}_k z$. For this reason the branch points 0 and ∞ are in this case called *logarithmic branch points*, or *branch points of infinite multiplicity*.

Domains of a more general type than G in which it is possible to isolate the branches of $\mathrm{Ln}\, z$ can be obtained by taking a Jordan arc Γ' in the z-plane connecting point $z = 0$ with point $z = \infty$. The image curves in the w-plane are obtained through the mapping $w = \mathrm{Ln}\, z$ and are Jordan arcs γ_k' $(k = 0, \pm 1, \pm 2, \ldots)$ that separate the w-plane into an infinite number of curvilinear strips

g'_k with boundaries that are pairs of curves, say γ'_k and γ'_{k+1}. In domain G' whose boundary is Γ' we arrive at an infinite (denumerable) set of branches of $\operatorname{Ln} z$, i.e. $(\operatorname{Ln} z)_k$, each of which maps G' in a one-to-one manner into the corresponding domain g'_k.

Any function $(\operatorname{Ln} z)_k$ can be obtained from another function $(\operatorname{Ln} z)_m$ by adding an integral multiple of $2\pi i$.

For the derivative of $(\operatorname{Ln} z)_k$ we have the same formula

$$(\operatorname{Ln} z)'_k = \frac{1}{z}.$$

Since this result is independent of the choice of the branch of $\operatorname{Ln} z$, we can in general write

$$(\operatorname{Ln} z)' = \frac{1}{z},$$

where the left-hand side is understood to be the derivative of an arbitrary branch of $\operatorname{Ln} z$ isolated in the domain with the given point z.

3.20

THE POWER AND EXPONENTIAL FUNCTIONS: THE GENERAL CASE

In this section we will consider the general case of power and exponential functions and the logarithm with arbitrary base. As a preliminary we clarify the notion of a power function with an arbitrary exponent.

Suppose that a is an arbitrary nonzero number. If n is an integer, we know that a^n is defined as

$$a^n = |a|^n [\cos (n \operatorname{Arg} a) + i \sin (n \operatorname{Arg} a)].$$

If r is an arbitrary rational number equal to p/q, where q is a positive integer and p/q is irreducible, then a^r has q different values found from the formula derived in Sec. 1.2:

$$a^{p/q} = |a|_+^{p/q} \left[\cos \left(\frac{p}{q} \operatorname{Arg} a\right) + i \sin \left(\frac{p}{q} \operatorname{Arg} a\right)\right].$$

This formula includes the case of an integral exponent.

Now let us suppose that ρ is a real irrational number. We fix an arbitrary value of $\varphi = \operatorname{Arg} a$ and consider a sequence of rational numbers r_n converging to ρ. The sequence of definite values a^{r_n} equal to

$$|a|_+^{r_n} [\cos (r_n \operatorname{Arg} a) + i \sin (r_n \operatorname{Arg} a)],$$

obviously, converges to a definite limit

$$|a|_+^{\rho} [\cos (\rho \operatorname{Arg} a) + i \sin (\rho \operatorname{Arg} a)],$$

which we will take as one of the values of a^{ρ}. To find all the values of a^{ρ} with ρ irrational we must make Arg a run through all its values.

Since two different values of ρ Arg a differ by a term of the type $2k\rho\pi$, which cannot be divisible by 2π (k is a nonzero integer and ρ an irrational number), all values of a^{ρ} corresponding to different values of Arg a are different.

Hence, we have defined a^{α} for the case where α is an arbitrary real number. All values of a^{α} are given by the formula

$$a^{\alpha} = |a|_+^{\alpha} [\cos (\alpha \operatorname{Arg} a) + i \sin (\alpha \operatorname{Arg} a)]. \tag{3.54}$$

We have one value when α is an integer, several values when α is a rational number represented by an irreducible fraction p/q (exactly q values), and, finally, an infinite (denumerable) set of different values when α is an irrational number.

To determine the meaning of a^{α} when the exponent α is an arbitrary complex number, we note that (3.54) can be written

$$a^{\alpha} = e^{\alpha \ln |a|} [\cos (\alpha \operatorname{Arg} a) + i \sin (\alpha \operatorname{Arg} a)]$$
$$= \exp (\alpha \ln |a| + i\alpha \operatorname{Arg} a) = \exp (\alpha \operatorname{Ln} a).$$

The right-hand side of this formula has meaning not only for α real but for any complex value of α. Accordingly, we assume

$$a^{\alpha} = \exp (\alpha \operatorname{Ln} a) \tag{3.55}$$

to be valid for any complex number α.

Obviously, for α imaginary all values of a^{α} corresponding to different values of Ln a or, which is the same, to different values of Arg a are also different. Indeed, two different values of α Ln a differ by a number of the type $2\pi i\alpha$, which for α imaginary cannot be divisible by $2\pi i$.

Note that a^{α} with an arbitrary exponent α does not, in general, obey the addition rule for exponents or the rule by which exponents are multiplied when a^{α} is raised to another power. Namely, if $\alpha_1 (1 - k) - \alpha_2 k$ is not an integer for any integer k, then

$$a^{\alpha_1} a^{\alpha_2} = \exp (\alpha_1 \operatorname{Ln} a) \exp (\alpha_2 \operatorname{Ln} a) = \exp (\alpha_1 \operatorname{Ln} a + \alpha_2 \operatorname{Ln} a)$$
$$\neq \exp [(\alpha_1 + \alpha_2) \operatorname{Ln} a] = a^{\alpha_1 + \alpha_2}.$$

Similarly, if $\beta - k\alpha\beta$ is not an integer for any integer k, then

$$(a^{\alpha})^{\beta} = [\exp (\alpha \operatorname{Ln} a)]^{\beta} = \exp [\beta (\alpha \operatorname{Ln} a + 2k\pi i)] \neq \exp (\beta\alpha \operatorname{Ln} a)$$
$$= a^{\alpha\beta}.$$

Let us clarify the above reasoning by the following examples.

(1) $$1^{\sqrt{2}} = \exp(\sqrt{2}\,\mathrm{Ln}\,1) = \exp(2k\pi i\sqrt{2}) = \cos(2k\pi\sqrt{2}) + i\sin(2k\pi\sqrt{2})$$

$$(k = 0, \pm 1, \pm 2, \ldots),$$

(2) $$e^z = \exp(z\,\mathrm{Ln}\,e) = \exp[z(1 + 2k\pi i)] = \exp z \exp(2k\pi iz)$$

$$(k = 0, \pm 1, \pm 2, \ldots).$$

We see that only one value of e^z coincides with $\exp z$. The other values are $\exp z \exp 2\pi iz$, $\exp z \exp(-2\pi iz)$, etc. In particular, only one value of e^x (x real) coincides with the real positive number $\exp x$. The other values are $\exp x \exp 2\pi ix$, $\exp x \exp(-2\pi ix)$, etc. Overall there are a finite number of these values for x rational and an infinite number of values for x irrational. Nevertheless, we use the customary notation e^z as coinciding with $\exp z$. This use of a multiple-valued symbol is similar to the use of $\sqrt[n]{a}$ (a is a real positive number) as the only real positive value of the root.

(3) $$i^i = \exp(i\,\mathrm{Ln}\,i) = \exp\left[i\left(\frac{\pi}{2}i - 2k\pi i\right)\right] = \exp\left[(4k-1)\frac{\pi}{2}\right] = e^{(4k-1)\frac{\pi}{2}}$$

$$(k = 0, \pm 1, \pm 2, \ldots).$$

We see that all the values of i^i are positive real numbers, among which there are arbitrarily small and arbitrarily large ones.

Relying on the above definitions, we can now discuss the two multiple-valued functions

$$z^\alpha \text{ and } a^z,$$

where the first (α is an arbitrary complex number) is defined only for $z \neq 0$.

If α is an integer, z^α represents a rational function of a special type. It is well-defined then for $z = 0$, too, where it either vanishes ($\alpha > 0$) or has a pole ($\alpha < 0$). When α is a rational noninteger, i.e. $\alpha = p/q$ (q a positive integer and p/q an irreducible fraction), z^α can be represented in the form

$$z^\alpha = \sqrt[q]{z^p}.$$

This is a multiple-valued (q-valued) function, for which only $z = 0$ and $z = \infty$ are branch points of degree of ramification $q - 1$. In any region G obtained from the extended complex plane by connecting the branch points by a Jordan arc we can isolate q different differentiable branches of this function. These branches continuously

pass into each other as point z traverses curves that enclose the origin of coordinates or the point at infinity.

Finally, when α is not a rational number (i.e. α is an irrational or an imaginary number), z^α is infinitely valued. All its values are represented by the formula

$$z^\alpha = \exp(\alpha \operatorname{Ln} z).$$

Points $z = 0$ and $z = \infty$ are still branch points of this function but of infinite multiplicity (or of infinite degree of ramification), i.e. they are logarithmic branch points.

Indeed, as point $z = 0$ is circled once in the positive direction, the value of $\operatorname{Arg} z$ continuously changes and increases by 2π; the value of $\alpha \operatorname{Ln} z$, therefore, increases by $2\pi i\alpha$ and z^α acquires a factor $\exp(2\pi i\alpha) \neq 1$.

Let us now turn to the general exponential function a^z $(a \neq 0)$. For any finite value of z it can be represented in the form

$$a^z = \exp(z \operatorname{Ln} a).$$

To isolate a definite branch we must fix only one value, say $\operatorname{Ln} a = b$.

Assuming this, we arrive at a single-valued and differentiable function $\exp(bz)$. If now we take all possible values of $\operatorname{Ln} a$, we obtain all possible branches of a^z. Since two values of $\operatorname{Ln} a$ differ by a term of the type $2k\pi i$, any two branches of a^z differ by a factor of the type $\exp(2k\pi iz)$; this factor is a single-valued and everywhere differentiable function, which admits the value 1 only for real integral values of z. However, in this case the branches of the multiple-valued function differ considerably in their nature from the branches of all the multiple-valued functions studied earlier. Indeed, in all previous examples there existed such points in the extended complex plane (branch points) that by moving around them along Jordan curves and making the values of the function (a particular branch) change continuously, we were able to pass continuously from one branch to another. Here this is impossible simply because each branch represents a function that is continuous and single-valued in the entire finite plane. Irrespective of the Jordan curve we choose to traverse, after returning to the initial point we arrive at the same number z (even with another value of the argument) and, hence, at the same value of the function $\exp(bz)$ (b is the fixed value of $\operatorname{Ln} a$).

In this way the multiple-valued function a^z has not a single branch point and its continuous branches cannot transform into each other continuously. This enables us to view these functions as independent functions not linked to each other and single-valued

and differentiable in the entire plane, i.e. entire functions

$$\exp(z \ln a),\ \exp[z(\ln a + 2\pi i)],\ \exp[z(\ln a - 2\pi i)],\ \ldots .$$

The fact that all these different entire functions can be viewed as the branches of one infinitely valued function a^z has no more significance than the fact that $\sin z$ and $-\sin z$ can be viewed as the branches of the two-valued function $\sqrt{1-\cos^2 z}$ or that $\sinh z$ and $\cosh z$ can be viewed as the branches of the two-valued function $\frac{1}{2}[\exp z + \sqrt{\exp(-2z)}]$. (The reader will note, of course, that both $\sqrt{1-\cos^2 z}$ and $\frac{1}{2}[\exp z + \sqrt{\exp(-2z)}]$, just as the function a^z, have no branch points.)

Fixing one of the branches of the function $z = a^w = \exp(bw)$, where b is one of the values of $\mathrm{Ln}\, a$, we can study the function that is the inverse with respect to this branch. We find that

$$w = \frac{1}{b} \mathrm{Ln}\, z \quad (b = \ln a + 2k_0 \pi i). \tag{3.56}$$

This function differs from $\mathrm{Ln}\, z$ only in the constant factor $1/b$. Since from (3.56) it follows that

$$z = \exp(bw) = a^w \quad \text{(one of the values of } a^w\text{)},$$

we can consider w as the *logarithm of z to the base a*.

Thus, we define the logarithm of a complex number to the base a (a is a nonzero complex number) by the formula

$$\mathrm{Log}_a\, z = \frac{\mathrm{Ln}\, z}{\mathrm{Ln}\, a}, \tag{3.56'}$$

where the denominator is one of the infinite number of values of $\mathrm{Ln}\, a$ (the same value of b for all values of z). The definition requires, therefore, that we not only choose a definite value of a for base but fix one of the values of $\mathrm{Ln}\, a$ as well.

Here are some examples.

(1) $a = e$. If we fix $\mathrm{Ln}\, e$ as unity, then

$$\mathrm{Log}_e\, z = \mathrm{Ln}\, z,$$

which is the usual definition of the *natural* (or *Napierian*) *logarithm*. But if we take the value of $\mathrm{Ln}\, e$ to be $1 + 2\pi i$, then

$$\mathrm{Log}_e\, z = \frac{\mathrm{Ln}\, z}{1 + 2\pi i}.$$

The reader can easily verify that, with such a definition of all real positive numbers only those of the type e^k (k an integer) have a single real value of the natural logarithm.

(2) $a = 10$. Taking the value of Ln 10 equal to 2.302 585... = = 1/0.434 29... = $1/M$, we find that

$$\text{Log}_{10} z = M \,\text{Ln}\, z = 0.434\,29 \ldots \,\text{Ln}\, z.$$

This definition of the *common logarithm* of an arbitrary complex number z ($z \neq 0$) is consistent with the usual definition of the common logarithm of a real positive number. Namely, if $z = x > 0$, then taking the principal value of the logarithm we have

$$\log_{10} x = 0.434\,29 \ldots \ln x.$$

(3) $a = 1$. In this case, when defining the logarithm to base unity by formula (3.56′), we cannot use the principal value of Ln 1 equal to 0. Let us take the principal value of Ln 1 equal to $2\pi i$. Then by definition

$$\text{Log}_1 z = \frac{\text{Ln}\, z}{2\pi i} = \frac{1}{2\pi} \text{Arg}\, z - \frac{i}{2\pi} \ln |z|.$$

This implies that all the values of $\text{Log}_1 z$ are real numbers if $|z| = 1$ and imaginary numbers if $|z| \neq 1$. Consequently, only numbers represented by points on the unit circle have real logarithms to base unity. For complex numbers with moduli unity, $(1/2\pi)\,\text{Arg}\, z$ coincides with the logarithm to base unity we have just defined.

3.21

INVERSE TRIGONOMETRIC FUNCTIONS

In this section we will dwell briefly on the inverse trigonometric functions Arc cos z and Arc tan z. The function Arc cos z is defined via the equation

$$z = \cos w. \tag{3.57}$$

Substituting $\frac{\exp(iw) + \exp(-iw)}{2}$ for $\cos w$ and introducing the notation $\exp(iw) = t$, we can rewrite Eq. (3.57) as

$$z = \frac{t + t^{-1}}{2} \tag{3.58}$$

whence

$$t^2 - 2zt + 1 = 0 \tag{3.59}$$

and

$$t = z + \sqrt{z^2 - 1}. \tag{3.60}$$

Since both roots of Eq. (3.59) are nonzero (their product is unity), the equation

$$\exp(iw) = t$$

has roots (with respect to the unknown w). We find that

$$w=\frac{1}{i}\operatorname{Ln} t=\frac{1}{i}\operatorname{Ln}\left(z+\sqrt{z^2-1}\right). \tag{3.61}$$

(Note that we did all this already in Sec. 3.14.)

Thus, the multiple-valued function $w = \operatorname{Arc}\cos z$ is expressed in terms of the logarithm and the square root:

$$w=\operatorname{Arc}\cos z=\frac{1}{i}\operatorname{Ln}\left(z+\sqrt{z^2-1}\right). \tag{3.62}$$

There are mainly two branch points, ± 1. Indeed, if point z traverses once a Jordan curve that contains in its interior one of these points, one of the values of $\sqrt{z^2-1}$ goes into the other value, which differs from the first only in sign. In the process of traversal, the value $t = z + \sqrt{z^2-1}$, which is one root of the quadratic equation (3.59), is replaced by the other root of the equation equal to $1/t$ (we have already noted that the product of the two is unity). Hence, from the values $w = (1/i)\operatorname{Ln} t$ we go over to the values $w = (1/i)\operatorname{Ln}(1/t)$, which are obviously not equal to the initial values if $t \neq 1/t$. But t can be equal to $1/t$ only at $t = \pm 1$, and from (3.59) or (3.60) we can see that this is possible only if $z = \pm 1$. Since the Jordan curves that we use for traversal do not pass through $z = \pm 1$, we do not encounter this case and we can indeed state that the values of $w = \operatorname{Arc}\cos z$ change as the result of such traversals. The points $z = \pm 1$ are, therefore, branch points of $\operatorname{Arc}\cos z$. The presence of these branch points is due to the square root in (3.62). But (3.62) has the form $w = (1/i)\operatorname{Ln} t$ (with $t = z + \sqrt{z^2+1}$), and we expect that there are also branch points corresponding to the two branch points of $\operatorname{Ln} t$, i.e. $t = 0$ and $t = \infty$.

From Sec. 3.14 we know that to each simple (complete) traversal by point t of a circle with center at the origin of coordinates there corresponds a simple (complete) traversal by point z of an ellipse in the z-plane with foci at ± 1, and vice versa.

Therefore, to a simple traversal by point z of an ellipse with foci at ± 1 there corresponds a change in $\operatorname{Arg} t$ equal to $\pm 2\pi$ and hence a change in $(1/i)\operatorname{Ln} t$ also equal to $\pm 2\pi$. Since such an ellipse may belong to any preassigned neighborhood of the point $z = \infty$, this point is a logarithmic branch point for $\operatorname{Arc}\cos z$.

Of course, the traversal of any ellipse with its foci at ± 1 can be considered in the same way as we did the traversal of a Jordan curve in the neighborhood of point $z = 0$. But not a single ellipse of this type lies entirely in the neighborhood $|z| < \rho$, where $\rho \leqslant 1$.

Let us show that neither point $z = 0$ nor any other point of the extended z-plane except the points we discussed above ($z = \pm 1$ and $z = \infty$) can be a branch point for $\operatorname{Arc}\cos z$. Indeed, formula (3.60)

gives for $z = z_0$ two different values, t_0' and t_0'', that are not 0, ± 1, or ∞ and that satisfy the condition $t_0' t_0'' = 1$. We can take such small neighborhoods U' and U'' of these points that they do not contain 0 and ∞ and neither of them, U' or U'', separately contains points t_1 and t_2 such that $t_1 t_2 = 1$. Indeed, if point t_0' does not lie on the unit circle, say $|t_0'| < 1$, then point t_0'' does not lie on the unit circle either and $|t_0''| > 1$. In this case it suffices to choose U' and U'' in a way such that one (U') lies inside the unit circle and the other (U'') outside (Fig. 32). But if t_0' lies on the unit circle in, for

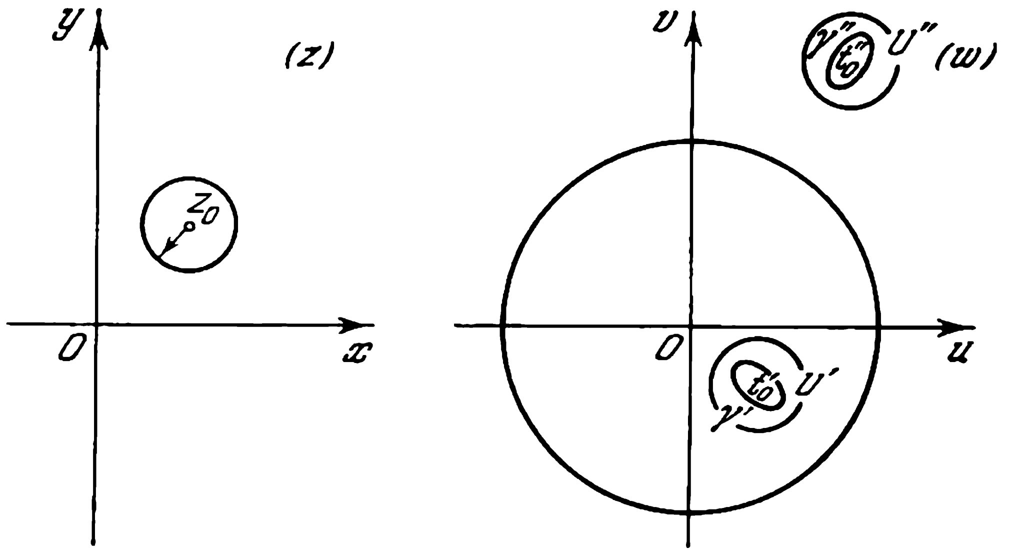

Fig. 32

instance, the upper half-plane, then $t_0'' = 1/t_0'$ lies in the lower half-plane and on the unit circle, too. In this case one neighborhood (U') can be taken in the upper half-plane and the other (U'') in the lower. Both in U' and in U'' the function $z = \frac{1}{2}(t + 1/t)$ is univalent (only in pairs of points that satisfy the relation $t't'' = 1$ does this function assume equal values) and, hence, maps U' and U'' in a one-to-one manner into domains g' and g'' in the z-plane that contain z_0 as their interior point (the image of centers t_0' and t_0'' of the neighborhoods U' and U'').

Now we take circles $|z - z_0| = \rho$ with centers at z_0 and so small that they belong both to g' and to g''. Obviously, all circles with sufficiently small radii satisfy this condition. Then by virtue of the mapping (3.58) each such circle has corresponding to it two Jordan curves γ' and γ'', one in U' and the other in U''. Not one of these closed curves encloses point O. Therefore, when z traverses a circle $|z - z_0| = \rho$, point t traverses either γ' (corresponding to one branch of (3.60)) or γ'' (corresponding to the other branch of (3.60)). If we fix the value of Arg t at some point of γ' (or γ'') prior to traversal, this value as a result of the traversal changes continuously

and returns to the initial value (because neither γ' nor γ'' contains point $t = 0$ in its interior). For this reason as a result of this traversal we will return to the initial value of $(1/i)$ Ln t = Arc cos z, too.

Thus, a point z_0 that differs from ± 1 and ∞ cannot be a branch point for Arc cos z.

To obtain a domain in the z-plane where we can isolate the continuous branches of Arc cos z, we must connect the branch points by a Jordan arc. For one, we can take the infinite segment Δ of the real axis that connects -1 and $+1$ through the point at infinity. This segment is the boundary of a domain G.

What we know of function (3.58) implies that this function maps into G in a one-to-one manner both upper and lower half-planes of the t-plane. On the other hand, the function $w = (1/i)$ Ln t maps each of them into strips in the w-plane parallel to the imaginary axis all of width π; namely, the upper half-plane is mapped into the strips g_{2k-1}:

$$(2k - 1)\pi < u < 2k\pi \quad (k = 0, \pm 1, \pm 2, \ldots),$$

and the lower half-plane into the strips g_{2k}:

$$2k\pi < u < (2k + 1)\pi.$$

Thus, the images of domain G in the w-plane are the strips g_n.

To fix a branch of Arc cos z in G it suffices to indicate to which of the strips g_n its values belong. In this manner we arrive at the branches: Arc $\cos_0 z$, Arc $\cos_1 z$, Arc $\cos_{-1} z$, ... However, it is enough to fix the value of Arc cos z at one point in G, e.g. at the origin of coordinates. Then the strip g_n into which this point is mapped determines the entire branch of Arc cos z.

The branches of Arc cos z can be defined, of course, in many other domains in the z-plane. One is domain G', whose boundary consists of the finite segment δ of the real axis connecting points -1 and 1 and the positive half of the imaginary axis. Another example is domain G'', whose boundary consists of the same segment δ and the negative half of the imaginary axis. The reader is advised to find the domains in the w-plane into which the corresponding branches of Arc cos z map G' and G''.

We turn to the function w = Arc tan z, which is the inverse of $z = \tan w$. Expressing $\tan w$ in terms of $\sin w$ and $\cos w$ and then in terms of the exponential function, we obtain

$$z = \frac{1}{i} \frac{e^{iw} - e^{-iw}}{e^{iw} + e^{-iw}} = \frac{1}{i} \frac{e^{2iw} - 1}{e^{2iw} + 1}, \tag{3.63}$$

or, assuming that $e^{2iw} = \tau$,

$$z = \frac{1}{i} \frac{\tau - 1}{\tau + 1},$$

whence

$$\tau = \frac{1+iz}{1-iz},$$

and, finally,

$$w = \frac{1}{2i} \operatorname{Ln} \frac{1+iz}{1-iz}.$$

Therefore, Arc tan z is expressed in terms of the logarithm of the linear-fractional function:

$$w = \operatorname{Arc} \tan z = \frac{1}{2i} \operatorname{Ln} \frac{1+iz}{1-iz}. \tag{3.64}$$

The reader is advised to check whether Arc tan z has only two branch points: $\pm i$. The simplest domains in which it is possible to isolate the branches of Arc tan z are (a) domain D, whose boundary is the infinite segment Δ of the imaginary axis connecting points $-i$ and $+i$ through the point at infinity, and (b) domain d, whose boundary is the finite segment δ connecting $-i$ and $+i$.

The reader will easily see that in D the branches of Arc tan z map D in a one-to-one manner and conformally into strips of width that are parallel to the imaginary axis,

$$k\pi - \frac{\pi}{2} < u < k\pi + \frac{\pi}{2} \quad (k = 0, \pm 1, \ldots),$$

while in d the branches map d into similar strips,

$$k\pi < u < (k+1)\pi.$$

4

SERIES WITH COMPLEX TERMS. POWER SERIES

4.1 CONVERGENT AND DIVERGENT SERIES

Let us suppose that $\{w_n = u_n + iv_n\}$ is a sequence of complex numbers. The expression

$$w_1 + w_2 + w_3 + \ldots + w_n + \ldots, \tag{4.1}$$

or, in brief form, $\sum\limits_1^\infty w_n$, is called a *series*, the numbers $w_1, w_2, u_3, \ldots$ are called *terms* of the series, and the sums $s_n = w_1 + w_2 + \ldots$ $\ldots + w_n$ $(n = 1, 2, \ldots)$ the *partial sums* of the series. If the sequence of partial sums converges, we say that the series *converges*, or *is convergent*, and the limit of the converging sequence $\lim\limits_{n\to\infty} s_n = s$ is called the *sum* of the series; in this case we write $\sum\limits_1^\infty w_n = s$. If the sequence $\{s_n\}$ is not convergent, we say that the series *diverges*, or *is divergent*. Obviously, these definitions include as particular cases the well-known definitions for series with real-valued terms. Noting that

$$s_n = \sum_1^n u_k + i \sum_1^n v_k$$

and that $\lim\limits_{n\to\infty} s_n = s$ is equivalent to two relationships,

$$\lim_{n\to\infty} \sum_1^n u_k = \operatorname{Re} s = \sigma \quad \text{and} \quad \lim_{n\to\infty} \sum_1^n v_k = \operatorname{Im} s = \tau,$$

we conclude that *a series with complex terms is convergent if and only if two series, one consisting of the real parts of the complex terms and the other of the imaginary parts, i.e.* $\sum\limits_1^\infty u_n$ *and* $\sum\limits_1^\infty v_n$, *converge.*

Applying the general Cauchy condition for the convergence of a sequence to $\{s_n\}$, we arrive at the following *general condition for*

the convergence of series: the series (4.1) is convergent if and only if for any given $\varepsilon > 0$ there exists an $N(\varepsilon)$ such that $|s_{n+p} - s_n| < \varepsilon$ for $n > N(\varepsilon)$ and any positive integer p. We find the *necessary condition for the convergence of a series* as a particular case of the above general condition ($p = 1$): $\lim\limits_{n\to\infty} w_{n+1} = 0$, i.e. $\lim\limits_{n\to\infty} w_n = 0$.

The series (4.1) is called *absolutely convergent* if the series $\sum\limits_1^\infty |w_n|$ is convergent. From the general condition for convergence we see that every absolutely convergent series converges. The converse, in general, is not true, which follows from the well-known example of a convergent, but not absolutely convergent, series with real terms, $\sum\limits_1^\infty (-1)^{n-1}\frac{1}{n}$.

The double inequalities

$$\left.\begin{matrix} |u_n| \\ |v_n| \end{matrix}\right\} \leqslant |w_n| \leqslant |u_n| + |v_n|$$

imply that the aboslute convergence of $\sum\limits_1^\infty w_n$ is equivalent to the convergence of $\sum\limits_1^\infty |u_n|$ and $\sum\limits_1^\infty |v_n|$, i.e. the absolute convergence of $\sum\limits_1^\infty u_n$ and $\sum\limits_1^\infty v_n$. Hence, to absolutely convergent series with complex terms we can apply the theorem stating that *any alteration of the order of terms in a series does not change the sum of the series.* We can verify whether a series of type (4.1) is absolutely convergent by applying any convergence criterion for series with positive terms (positive series), e.g. *Cauchy's criterion* (the series converges absolutely if $\sqrt[n]{|w_n|} \leqslant q < 1$ for $n \geqslant n_0$) or *D'Alembert's criterion* (the series converges absolutely if $|\omega_{n+1}| / |w_n| \leqslant q < 1$ for $n \geqslant n_0$).

We note, finally, that the sum or difference of two convergent series $\sum\limits_1^\infty w_n' = s'$ and $\sum\limits_1^\infty w_n'' = s''$ is expressed in terms of the convergent series

$$\sum_1^\infty (w_n' \pm w_n'') = s' \pm s''.$$

If, in addition, the two series are absolutely convergent, the series

$$\sum_1^\infty (w_1' w_n'' + w_2' w_{n-1}'' + \ldots + w_n' w_1'')$$

is absolutely convergent and its sum is $s's''$. The last statements can be proved for complex series in the same way as this is done for real series.

4.2
THE CAUCHY-HADAMARD THEOREM

The power series

$$a_0 + a_1 (z - z_0) + \ldots + a_n (z - z_0)^n + \ldots, \tag{4.2}$$

where $a_0, a_1, \ldots, a_n, \ldots$ are fixed complex numbers, and z is the complex variable, is the simplest example of a functional series (i.e. a series whose terms are functions of z). Generally speaking, a series of this type may converge for some values of z and diverge for others. The behavior of such series in this respect is regulated by

The Cauchy-Hadamard theorem. *If* $\overline{\lim} \sqrt[n]{|a_n|} = \Lambda$, *then at* $\Lambda = 0$ *the series* (4.2) *is absolutely convergent in the entire complex plane, at* $\Lambda = \infty$ *it converges only at point* $z = z_0$, *and, finally, at* $0 < \Lambda < \infty$ *it converges absolutely in the circle* K: $|z - z_0| < 1/\Lambda$ *and diverges outside this circle.**

Proof. (a) Suppose first that $\Lambda = 0$. Then $\overline{\lim} \sqrt[n]{|a_n|} = \lim \sqrt[n]{|a_n|} = 0$ and, hence, $\lim \sqrt[n]{|a_n| \, |z - z_0|^n} = 0$ for any z. In view of Cauchy's criterion, we find that the series $\sum\limits_0^\infty |a_n| \, |z - z_0|^n$, i.e. series (4.2) is absolutely convergent.

(b) Suppose that $\Lambda = \infty$. Then there exists a sequence of increasing subscripts $\{k_n\}$ such that $\sqrt[k_n]{|a_{k_n}|} \to \infty$. For this reason, for any $z \neq z_0$ we have $\sqrt[k_n]{|a_{k_n}| \, |z - z_0|^{k_n}} = \sqrt[k_n]{|a_{k_n}|} \, |z - z_0| \to \infty$, which means that $|a_{k_n}| \, |z - z_0|^{k_n} \to \infty$, i.e. series (4.2) does not meet the necessary condition for convergence.

(c) Suppose now that $0 < \Lambda < \infty$. If $z = z_0$, all terms in (4.2) starting from the second vanish and, hence, the series converges absolutely. If $z \neq z_0$ and point z lies inside the circle K, we can put $|z - z_0| = \theta'^2/\Lambda$, where $0 < \theta' < 1$. Since $\Lambda' = (\Lambda/\theta') > \Lambda$, in view of the properties of a superior limit there exists an N_1 such

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* The number Λ $(-\infty \leqslant \Lambda \leqslant +\infty)$ is known as the *limit superior* (or *superior limit*) of a sequence of real numbers $\{\alpha_n\}$, i.e. $\Lambda = \overline{\lim\limits_{n \to \infty}} \, \alpha_n$, if the following two conditions are met:

(1) for each $\Lambda' > \Lambda$ there exists an N such that $\alpha_n < \Lambda'$ for $n > N$;
(2) there is a subsequence $\{\alpha_{k_n}\}$ converging to Λ.

In the general calculus course it is proved that every sequence of numbers has a finite or infinite limit. (One such proof is given in G. M. Fikhtengol'ts, *A Course of Differential and Integral Calculus* [in Russian], vol. 1, Nauka, Moscow, 1966, item 42.)

that $\sqrt[n]{|a_n|}<\Lambda'$ for $n>N_1$. Then for $n>N_1$ we find that $\sqrt[n]{|a_n|\,|z-z_0|^n}<\Lambda'\,|z-z_0|=\frac{\Lambda}{\theta'}\frac{\theta'^2}{\Lambda}=\theta'<1$, whence from Cauchy's criterion there follows the absolute convergence of (4.2). If z lies outside K, then $|z-z_0|=\frac{1}{\Lambda\theta}$ $(0<\theta<1)$. In view of the properties of a superior limit there exists a sequence of increasing subscripts $\{k_n\}$ such that $\sqrt[k_n]{|a_{k_n}|}\to\Lambda$, whence $\sqrt[k_n]{|a_{k_n}|\,|z-z_0|^{k_n}}\to\Lambda\,|z-z_0|=\frac{1}{\theta}>1$. Hence, $|a_{k_n}|\,|z-z_0|^{k_n}\to$ $\to\infty$, i.e. the necessary condition for convergence of (4.2) is not met.

A circle with its center at z_0 and a radius $1/\Lambda$ inside which a power series is absolutely convergent and outside which it is divergent is called the *circle of convergence* of the series and the number $R=1/\Lambda$ the *radius of convergence*.

These definitions extend to the limiting cases: $\Lambda=0$ $(R=\infty)$ and $\Lambda=\infty$ $(R=0)$. In the first case the circle of convergence is the entire finite complex plane (its exterior is an empty set), and in the second it degenerates into point z_0 (its exterior is the finite plane without point z_0).

When $0<\Lambda<+\infty$, the *circumference of the circle of convergence*, $|z-z_0|=R=1/\Lambda$, is the curve at the points of which a power series may behave differently (with respect to its convergence).

Consider, for example, the series $\sum\limits_{1}^{\infty}\frac{z^n}{n^\alpha}$ (α a real number). Here $\Lambda=\overline{\lim\limits_{n\to\infty}}\sqrt[n]{\frac{1}{n^\alpha}}=\lim\limits_{n\to\infty} n^{-\frac{\alpha}{n}}=1$ and, hence, the radius of convergence $R=1$. On the circle $|z|=1$ we have: $|z^n/n^\alpha|=1/n^\alpha$; whence for $\alpha>1$ the series converges at all points on the circle while for $\alpha\leqslant 1$ it diverges. Now assume that $0<\alpha\leqslant 1$. Then at point $z=1$ the series diverges; we will discover that at all other points on the circle $|z|=1$ the series converges. Indeed, if $z=e^{i\theta}$ $(0<\theta<2\pi)$, then

$$\sum_{n+1}^{n+p}\frac{e^{ik\theta}}{k^\alpha}=e^{i(n+1)\theta}\left\{\sum_{j=0}^{p-2}\sigma_j\left[\frac{1}{(n+j+1)^\alpha}-\frac{1}{(n+j+2)^\alpha}\right]+\frac{\sigma_{p-1}}{(n+p)^\alpha}\right\},$$

where

$$\sigma_j=1+e^{i\theta}+\ldots+e^{ij\theta}=\frac{1-e^{i(j+1)\theta}}{1-e^{i\theta}}=e^{ij\frac{\theta}{2}}\frac{\sin\frac{j+1}{2}\theta}{\sin\frac{\theta}{2}}.$$

Since $|\sigma_j| \leqslant 1/\sin(\theta/2)$, we find that

$$\left|\sum_{n+1}^{n+p} \frac{e^{ik\theta}}{k^\alpha}\right| < \frac{1}{\sin\frac{\theta}{2}} \left\{ \sum_{j=0}^{p-2} \left[\frac{1}{(n+j+1)^\alpha} - \frac{1}{(n+j+2)^\alpha} \right] + \frac{1}{(n+p)^\alpha} \right\}$$

$$= \frac{1}{\sin\frac{\theta}{2}(n+1)^\alpha} \to 0, \quad n \to \infty.$$

Whence follows the convergence of the series at $|z| = 1$ and $z \neq 1$

Thus, a power series may converge at all points on the circumference of the circle of convergence, diverge at all points on the circumference, or converge at some points and diverge at other points on the circumference.

A direct corollary of the Cauchy-Hadamard theorem is

Abel's first theorem on power series. *If a power series* $\sum_0^\infty a_n (z - z_0)^n$ *converges at* $z_1 \neq z_0$, *it converges absolutely in the circle* $|z - z_0| < |z_1 - z_0|$. Indeed, the hypothesis implies that z_1 cannot lie outside the circle of convergence of the series. Therefore, it must lie either in the circle or on its circumference. In both cases the circle $|z - z_0| < |z_1 - z_0|$ belongs to the circle of convergence and, hence, the series converges absolutely.

4.3

ANALYTICITY OF THE SUM OF A POWER SERIES

Let us show that *in the circle of convergence* $|z - z_0| < R$ $(R > 0)$ *of the power series* $\sum_0^\infty a_n (z - z_0)^n$ *its sum* $f(z)$ *is an analytic function and the derivative* $f'(z)$ *can be found by differentiating the series termwise*: $f'(z) = \sum_1^\infty n a_n (z - z_0)^{n-1}$. Let us see whether the radius of convergence R' of the latter series coincides with R. It is quite obvious that it coincides with the radius of convergence of $\sum_0^\infty n a_n (z - z_0)^n$. But $\overline{\lim\limits_{n\to\infty}} \sqrt[n]{n|a_n|} = \lim\limits_{n\to\infty} n^{\frac{1}{n}} \overline{\lim\limits_{n\to\infty}} \sqrt[n]{|a_n|} = \overline{\lim\limits_{n\to\infty}} \sqrt[n]{|a_n|}$; whence

$$R' = \left(\overline{\lim_{n\to\infty}} \sqrt[n]{n|a_n|}\right)^{-1} = \left(\overline{\lim_{n\to\infty}} \sqrt[n]{|a_n|}\right)^{-1} = R.$$

Suppose that z_1 is a point in the circle $|z - z_0| < R$. We take a point ζ such that $|z_1 - z_0| < |\zeta - z_0| = \rho < R$. Since the

series $\sum_{1}^{\infty} na_n (\zeta - z_0)^{n-1}$ is absolutely convergent, for any $\varepsilon > 0$ the inequality

$$\sum_{n+1}^{\infty} j \mid a_j \mid \rho^{j-1} < \varepsilon$$

is valid for $n \geqslant n_0(\varepsilon) = n_0$. Assuming that $\sum_{1}^{\infty} na_n (z - z_0)^{n-1} = \varphi(z)$, we see that at $z \neq z_1$ and $\mid z - z_0 \mid \leqslant \rho$ we have

$$\left| \frac{f(z) - f(z_1)}{z - z_1} - \varphi(z_1) \right|$$

$$= \left| \sum_{0}^{\infty} a_n \frac{(z - z_0)^n - (z_1 - z_0)^n}{z - z_1} - \sum_{1}^{\infty} na_n (z_1 - z_0)^{n-1} \right|$$

$$= \Bigg| \sum_{1}^{\infty} a_n [(z - z_0)^{n-1} + (z - z_0)^{n-2} (z_1 - z_0) + \ldots + (z_1 - z_0)^{n-1}]$$

$$- \sum_{1}^{\infty} na_n (z_1 - z_0)^{n-1} \Bigg| \leqslant \Bigg| \sum_{1}^{n_0} a_n [(z - z_0)^{n-1}$$

$$+ \ldots + (z_1 - z_0)^{n-1} - n (z_1 - z_0)^{n-1}] \Bigg| + 2 \sum_{n_0+1}^{\infty} n \mid a_n \mid \rho^{n-1}.$$

The first term (inside the bars) tends to zero as $z \to z_1$ and, therefore, is less than ε for $\mid z - z_1 \mid < \delta(\varepsilon)$; the second term is less than 2ε because of the choice of n_0. Hence,

$$\left| \frac{f(z) - f(z_1)}{z - z_1} - \varphi(z_1) \right| < 3\varepsilon \text{ for } |z - z_1| < \delta(\varepsilon),$$

from which follows the validity of the theorem.

Applying this theorem to the sum of the series $\sum_{1}^{\infty} na_n (z - z_0)^{n-1} = f'(z)$, we find that $f'(z)$ is also an analytic function in the circle $\mid z - z_0 \mid < R$ and that $f''(z) = \sum_{2}^{\infty} n(n - 1) a_n (z - z_0)^{n-2}$. Successive application of this theorem yields the following result:

The sum of the power series $\sum_{0}^{\infty} a_n (z - z_0)^n = f(z)$ *is infinitely differentiable in the circle of convergence* $\mid z - z_0 \mid < R$ $(R > 0)$.

The derivative of order p can be obtained by termwise differentiation of the series p times:

$$f^{(p)}(z)=\sum_{0}^{\infty} n(n-1)\ldots(n-p+1)a_n(z-z_0)^{n-p}$$

$$=\sum_{n=p}^{\infty} n(n-1)\ldots(n-p+1)a_n(z-z_0)^{n-p}.$$

If we assume now that $z = z_0$, we find that

$$f^{(p)}(z_0)=[p(p-1)\times\ldots\times 1]\,a_p,$$

whence

$$a_p=\frac{f^{(p)}(z_0)}{p!}.$$

This result remains valid for $p = 0$ if we assume that $f^{(0)}(z) = f(z)$ and $0! = 1$. We can then rewrite the power series as

$$f(z)=\sum_{0}^{\infty}\frac{f^{(n)}(z_0)}{n!}(z-z_0)^n.$$

The series is called the *Taylor series* of $f(z)$. Therefore, we have proved the

Theorem. *Each power series with a positive radius of convergence is a Taylor series of its sum.*

Let us suppose that $\sum_{0}^{\infty} a_n(z-z_0)^n$ and $\sum_{0}^{\infty} b_n(z-z_0)^n$ have the same sum in the neighborhood $|z-z_0|<\rho\ (\rho>0)$ of point z_0. Then by the above theorem we have

$$a_n=b_n=\frac{f^{(n)}(z_0)}{n!}\quad(n=0, 1, 2, \ldots),$$

i.e. the corresponding coefficients of these series are equal. This reflects the *uniqueness of the expansion into a power series*.

As an example that illustrates the above property we establish the form of power series in z that represent even or odd functions of z.

We assume that the function

$$f(z) = a_0 + a_1z + a_2z^2 + \ldots + a_nz^n + \ldots$$

is *even*, i.e. $f(-z) = f(z)$. Then the sum of the given series coincides with the sum of the series

$$f(-z) = a_0 - a_1z + a_2z^2 + \ldots + (-1)^na_nz^n + \ldots,$$

whence according to the last theorem

$$a_n = (-1)^na_n\ (n = 0, 1, 2, \ldots).$$

For n odd we find that $a_n = -a_n$, whence $a_n = 0$ $(n = 1, 3, 5, \ldots)$ and, therefore,

$$f(z) = a_0 + a_2 z^2 + \ldots + a_{2m} z^{2m} + \ldots .$$

In a similar manner we can show that if $f(z)$ is an *odd* function, i.e. $f(-z) = -f(z)$, all coefficients with even subscripts vanish and, hence,

$$f(z) = a_1 z + a_3 z^3 + \ldots + a_{2m-1} z^{2m-1} + \ldots .$$

We have proved in this section that the sum of every power series is an analytic function. Later on we will be able to prove that each analytic function can be expanded in a power series in a neighborhood of each point of analyticity. This will enable us to extend the properties of sums of power series established in this section onto analytic functions.

4.4 UNIFORMLY CONVERGENT SERIES

The notion of uniform convergence of series with real terms and the properties of such series are extended onto series with complex terms. Namely, the series $\sum_1^\infty f_j(z)$, where the $f_j(z)$ are defined on a set E, is said to be *uniformly convergent* (or *to converge uniformly*) on a set $E_1 \subset E$ if it converges on E_1 and if for every positive ε the inequality $\left|\sum_{n+1}^\infty f_j(z)\right| < \varepsilon$ is valid at all points of E_1 for $n > N(\varepsilon)$.

Assuming that $\sum_1^\infty f_j(z) = f(z)$, $\sum_1^n f_j(z) = S_n(z)$, and $\sum_{n+1}^\infty f_j(z) = f(z) - S_n(z) = R_n(z)$, we can rewrite this condition as

$$\sup_{z \in E_1} |R_n(z)| \leqslant \varepsilon \text{ for } n > N(\varepsilon), \text{ or } \lim_{n \to \infty} \sup_{z \in E_1} |R_n(z)| = 0.$$

In the case where the series converges on E_1 but not uniformly we have $\lim\limits_{n\to\infty} R_n(z) = 0$, $z \in E_1$, but $\lim\limits_{n\to\infty} \sup\limits_{z \in E_1} |R_n(z)|$ either does not exist or is nonzero.

The condition for uniform convergence can be written as follows: $\left|\sum_{n+1}^{n+p} f_j(z)\right| < \varepsilon$ at all points of E_1 for $n \geqslant N(\varepsilon)$ and any positive integer p. From this follows the *criterion for uniform convergence* (*Weierstrass's criterion for uniform convergence*): if the series $\sum_1^\infty u_n$ with constant positive terms converges and $|f_n(z)| \leqslant u_n$, $z \in E_1$, $n \geqslant N_1$, then the series $\sum_1^\infty f_n(z)$ is uniformly absolutely convergent.

Let us apply this criterion to the power series $\sum_{0}^{\infty} a_n (z - z_0)^n$ whose radius of convergence is $R > 0$. We will see that in each circle $| z - z_0 | \leqslant r$, where $r < R$, it converges uniformly.

Let us suppose that z_1 is a point in the circle of convergence but lying outside the circle $| z - z_0 | \leqslant r$. Then $r < | z_1 - z_0 | = \rho < R$ and, hence, the inequality $| a_n (z - z_0)^n | \leqslant | a_n (z_1 - z_0)^n |$ is valid at each point of the circle $| z - z_0 | \leqslant r$. But a series with constant terms $| a_n (z_1 - z_0)^n |$ converges; hence, the series $\sum_{0}^{\infty} a_n (z - z_0)^n$ is uniformly convergent in the circle $| z - z_0 | \leqslant r$ $(r < R)$.

Let us show that there are power series that converge but not uniformly in the circle of convergence. The simplest example here is the geometric series $\sum_{0}^{\infty} z^n$. In this case $R = 1$, $S_n (z) = 1 + z + \ldots + z^n = \frac{1 - z^{n+1}}{1 - z}$, $f (z) = \frac{1}{1 - z}$, $R_n (z) = \frac{z^{n+1}}{1 - z}$ and $\sup\limits_{|z|<1} |R_n (z)| = \sup\limits_{|z|<1} \frac{|z|^{n+1}}{|1 - z|} = +\infty$ for any n, i. e. the criterion for uniform convergence is not met.

On the other hand, there are power series that converge uniformly in the closed circle of convergence. An example is the series $\sum_{1}^{\infty} z^n/n^2$. Here $R = 1$ and $| z^n/n^2 | \leqslant 1/n^2$ for $| z | \leqslant 1$. By Weierstrass's criterion for uniform convergence the series converges uniformly in the closed unit circle.

Note, finally, that a well-known theorem is carried over without variations onto uniformly convergent series of complex functions; namely, *if the functions $f_j (z)$ are continuous on E and the series $\sum_{1}^{\infty} f_j (z)$ is uniformly convergent on E, the sum of the series, $f (z)$, is continuous on E, too.*

5

INTEGRATING FUNCTIONS OF A COMPLEX VARIABLE

5.1

AN INTEGRAL WITH RESPECT TO A COMPLEX VARIABLE

Let us suppose that L is a rectifiable curve in the complex plane and $z = \lambda(t) = x(t) + iy(t)$ $(\alpha \leqslant t \leqslant \beta)$ is its equation.* On L we can choose one of two directions, one corresponding to increasing t's and the other to decreasing t's. In what follows we are assuming that L denotes a curve with a chosen direction on it; we denote the same curve but with the opposite direction by $-L$. For the sake of definiteness we choose the direction with increasing t's. Then we call point $z_0 = \lambda(\alpha)$ the *initial point* (or *lower end*) of L and $z' = \lambda(\beta)$ the *terminal point* (or *upper end*) of L. To each system of values of t, i.e. $t_0 = \alpha < t_1 < t_2 < \ldots < t_n = \beta$, there corresponds a subdivision T of L into arcs $l_0, l_1, \ldots, l_{n-1}$, where point $z_k = \lambda(t_k) = x_k + iy_k$ is the lower end of arc l_k and $z_{k+1} = \lambda(t_{k+1})$ $(z_n = z')$ its upper end. If $w = f(z) = u(x, y) + iv(x, y)$ is single-valued and continuous on L, then by choosing in each segment $t_k \leqslant t \leqslant t_{k+1}$ one value of t equal to τ_k we have one point $\zeta_k = \lambda(\tau_k)$ on each arc l_k. At such points we find the values of $f(z)$ and build the complex Riemann sum $\sum\limits_0^{n-1} f(\zeta_k) \times (z_{k+1} - z_k)$ corresponding to the given subdivision T. Assuming for the sake of brevity that $f(\zeta_k) = u(\xi_k, \eta_k) + iv(\xi_k, \eta_k) = u_k + iv_k$ and $z_{k+1} - z_k = (x_{k+1} - x_k) + i(y_{k+1} - y_k) = \Delta x_k + i\Delta y_k$, we obtain

$$\sum_0^{n-1} f(\zeta_k)(z_{k+1} - z_k) = \sum_0^{n-1} (u_k \Delta x_k - v_k \Delta y_k) + i \sum_0^{n-1} (v_k \Delta x_k + u_k \Delta y_k).$$

Obviously, the real and imaginary parts of the complex Riemann sum are Riemann sums for a pair of real-valued functions of x and y, introduced in integral calculus to define the curvilinear integral

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* For the definition of a rectifiable curve, see, for instance, S. M. Nikolsky, *A Course of Mathematical Analysis*, vol. 1, Mir Publishers, Moscow, 1977 (reprinted in 1981), Sec. 6.7.

of the type $\int\limits_L P(x, y)\,dx + Q(x, y)\,dy$; the first sum is built for the pair of functions $u(x, y)$ and $-v(x, y)$ and the given subdivision T and the second for the pair $v(x, y)$ and $u(x, y)$ and the same subdivision. Since each of the functions in the two pairs is continuous on L and the curve is rectifiable, by a well-known theorem from calculus* these sums tend to finite limits $\int\limits_L u(x, y)\,dx - v(x, y)\,dy$ and $\int\limits_L v(x, y)\,dx + u(x, y)\,dy$ for any sequence of subdivisions T that satisfy the condition

$$\delta_T = \max(|t_1 - t_0|, |t_2 - t_1|, \ldots, |t_n - t_{n-1}|) \to 0.$$

These limits, as is known, do not depend on the sequence of subdivisions and the choice of points $\zeta_0, \zeta_1, \ldots, \zeta_{n-1}$ on the arcs of each subdivision. Therefore, there exists the limit

$$\lim_{\delta_T \to 0} \sum_{k=0}^{n-1} f(\zeta_k)(z_{k+1} - z_k) = \int\limits_L u(x, y)\,dx - v(x, y)\,dy + i\int\limits_L v(x, y)\,dx + u(x, y)\,dy,$$

which is called the *integral of the function* $f(z)$ *along the curve* L (in the chosen direction) and is denoted by $\int\limits_L f(z)\,dz$:

$$\lim_{\delta_T \to 0} \sum_{k=0}^{n-1} f(\zeta_k)(z_{k+1} - z_k) = \int\limits_L f(z)\,dz.$$

Here L is called the *contour* (or *path*) *of the integration.*

From the above definition of the integral it follows that the integrals of $f(z)$ along L and along $-L$ are linked through the relation

$$\int\limits_{-L} f(z)\,dz = -\int\limits_L f(z)\,dz.$$

Examples. (a) Suppose that $f(z) = 1$. Then the Riemann sum

$$\sum_{k=0}^{n-1} f(\zeta_k)(z_{k+1} - z_k) = \sum_{k=0}^{n-1} (z_{k+1} - z_k) = z_n - z_0 = z' - z_0,$$

* A proof of this theorem is given in many textbooks on calculus. See, for instance, G. M. Fikhtengol'ts, *A Course of Differential and Integral Calculus* [in Russian], vol. 3, Nauka, Moscow, 1966, item 585.

whence $\int_L dz = z' - z_0$. In particular, when L is a closed curve, i.e. $z' = z_0$, the integral $\int_L dz = 0$.

(b) Suppose that $f(z) = z$. Then, assuming that $\zeta_k = z_k$ $(k = 0, 1, 2, \ldots, n-1)$, we find the Riemann sums $\sum_{k=0}^{n-1} z_k (z_{k+1} - z_k)$. If we put $\zeta_k = z_{k+1}$ $(k = 0, 1, 2, \ldots, n-1)$, the Riemann sum is $\sum_{k=0}^{n-1} z_{k+1}(z_{k+1} - z_k)$. Since both sums have the same limit, their arithmetic mean has the same limit $\int_L z\,dz$:

$$\int_L z\,dz = \lim_{\delta \to 0} \frac{1}{2} \sum_{k=0}^{n-1} (z_{k+1} + z_k)(z_{k+1} - z_k)$$

$$= \lim_{\delta \to 0} \frac{1}{2} \sum_{k=0}^{n-1} (z_{k+1}^2 - z_k^2) = \lim_{\delta \to 0} \frac{1}{2} (z_n^2 - z_0^2) = \frac{1}{2} (z'^2 - z_0^2).$$

If L is a closed curve, we again find that $\int_L z\,dz = 0$.

(c) Let us now evaluate the integral $\int_{|z-a|=\rho} \frac{dz}{z-a}$, where the circle $|z-a| = \rho$ is traversed counterclockwise. We can write the equation of the circle as $z = a + \rho e^{it}$ $(0 \leqslant t \leqslant 2\pi)$. We choose the subdivision $t_k = 2k\pi/n$ and also put $\tau_k = \frac{t_k + t_{k+1}}{2} = \frac{2k+1}{n}\pi$ $(k = 0, 1, 2, \ldots, n-1)$; then $z_k = a + \rho e^{\frac{2k\pi i}{n}}$ and $\zeta_k = a + \rho e^{\frac{(2k+1)\pi i}{n}}$. The corresponding Riemann sums are

$$\sum_0^{n-1} \frac{1}{\rho e^{(2k+1)\frac{\pi i}{n}}} \rho \left(e^{\frac{2(k+1)\pi i}{n}} - e^{\frac{2k\pi i}{n}}\right) = \sum_0^{n-1} \left(e^{\frac{\pi i}{n}} - e^{-\frac{\pi i}{n}}\right)$$

$$= n\left(e^{\frac{\pi i}{n}} - e^{-\frac{\pi i}{n}}\right) = n2i \sin \frac{\pi}{n},$$

whence

$$\int_{|z-a|=\rho} \frac{dz}{z-a} = \lim_{n \to \infty} 2ni \sin \frac{\pi}{n} = 2\pi i.$$

5.2
BASIC PROPERTIES OF INTEGRALS

The properties of the integral $\int_L f(z)\,dz$ follow, as corollaries, from the properties of curvilinear integrals of the type $\int_L P(x, y)\,dx + Q(x, y)\,dy$. Below we give these properties.

(i) $$\int_L \left(\sum_{k=1}^{p} c_k f_k(z)\right) dz = \sum_{k=1}^{p} c_k \int_L f_k(z)\,dz.$$

(ii) $$\int_{L_1+L_2+\ldots+L_q} f(z)\,dz = \sum_{j=1}^{q} \int_{L_j} f(z)\,dz.$$

Here by $L_1 + L_2 + \ldots + L_q$ we denote a curve composed of the arcs L_j in a way such that the upper end of L_j coincides with the lower end of L_{j+1} $(j = 1, 2, \ldots, q-1)$.

It often becomes necessary to estimate the modulus of the integral from above. For the modulus of the Riemann sum the *estimate* is

$$\left|\sum_{k=0}^{n-1} f(\zeta_k)(z_{k+1}-z_k)\right| \leqslant \sum_{k=0}^{n-1} |f(\zeta_k)|\,|z_{k+1}-z_k|.$$

It is obvious that $|z_{k+1} - z_k|$ is the length of the chord connecting points z_k and z_{k+1}; hence $|z_{k+1} - z_k| \leqslant$ (length of l_k) $= \sigma_k$ and, therefore,

$$\left|\sum_{k=0}^{n-1} f(\zeta_k)(z_{k+1}-z_k)\right| \leqslant \sum_{k=0}^{n-1} |f(\zeta_k)|\,\sigma_k.$$

oing over to the limit under the condition that $\delta_T = \max(|t_1 - t_0|, \ldots, |t_n - t_{n-1}|) \to 0$, we obtain

(iii) $$\left|\int_L f(z)\,dz\right| \leqslant \int_L |f(z)|\,d\sigma.$$

On the right-hand side we have a curvilinear integral of the type $\int_L f(\sigma)\,d\sigma$, where σ is the length of an arc reckoned from the lower end of L to an arbitrary point on L in the chosen direction. If $|f(z)| \leqslant M$ $(z \in L)$, there follows from the above estimate a simpler but less accurate estimate

(iv) $$\left|\int_L f(z)\,dz\right| \leqslant M \times (\text{length of } L).$$

In particular,

$$\text{(iv}')\qquad \left|\int_L f(z)\,dz\right| \leqslant \sup_L |f(z)| \times (\text{length of } L).$$

There is also the following *formula for termwise integration of series*:

$$\text{(v)}\qquad \int_L \sum_1^\infty f_n(z)\,dz = \sum_1^\infty \int_L f_n(z)\,dz.$$

This formula is valid, for instance, under the following conditions: the functions $f_n(z)$ $(n = 1, 2, 3, \ldots)$ are continuous on L and the series $\sum\limits_1^\infty f_n(z)$ is uniformly convergent on L. The proof is the same as with functions of a real variable. Examples known from the general calculus course show (in the particular case of L being a segment of the real axis and all functions $f_n(z) = f_n(x)$ assuming real values) that for (v) to be valid it is not sufficient for the series $\sum\limits_1^\infty f_n(z)$ to simply converge.

5.3
REDUCTION TO ORDINARY INTEGRALS

Suppose that L is a *smooth* curve, which means that it can be represented in the parametric form

$$z = \lambda(t) = x(t) + iy(t) \quad (\alpha \leqslant t \leqslant \beta)$$

so that $\lambda(t)$ has a continuous derivative $\lambda'(t) = x'(t) + iy'(t)$ does not vanish on the interval $[\alpha, \beta]$. A geometrically smooth curve is characterized by the fact that at each point there exists a tangent whose angle of slope to the real axis (equal to Arg $\lambda'(t)$) continuously changes as the point of contact moves along the curve.

For the integral $\int\limits_L P(x, y)\,dx + Q(x, y)\,dy$ evaluated along L in the direction of increasing parameter there exists the following formula derived in the calculus course:

$$\int_L P(x, y)\,dx + Q(x, y)\,dy$$

$$= \int_\alpha^\beta \{P[x(t), y(t)]\,x'(t) + Q[x(t), y(t)]\,y'(t)\}\,dt,$$

which reduces the evaluation of a curvilinear integral to that of an ordinary integral. Therefore

$$\int_L f(z)\,dz = \int_L u\,dx - v\,dy + i\int_L v\,dx + u\,dy$$
$$= \int_\alpha^\beta \{u[x(t), y(t)]\,x'(t) - v[x(t), y(t)]\,y'(t)\}\,dt$$
$$+ i\int_\alpha^\beta \{v[x(t), y(t)]\,x'(t) + u[x(t), y(t)]\,y'(t)\}\,dt.$$

Note that the ordinary integral $\int_\alpha^\beta f(t)\,dt$ can be viewed as a particular case of the complex integral $\int_L f(z)\,dz$ defined in Sec. 5.1. Here L is a segment of the real axis that is traversed from α to β, and $f(z)$ assumes real values. Consequently, to the integrals on tbe right-hand side of the last equation we can apply property (i) of Sec. 5.2. This yields

$$\int_L f(z)\,dz = \int_\alpha^\beta \{u[x(t), y(t)]\,x'(t) - v[x(t), y(t)]\,y'(t)$$
$$+ iv[x(t), y(t)]\,x'(t) + iu[x(t), y(t)]\,y'(t)\}\,dt$$
$$= \int_\alpha^\beta \{u[x(t), y(t)] + iv[x(t), y(t)]\}\,[x'(t) + iy'(t)]\,dt.$$

Obviously, $u[x(t), y(t)] + iv[x(t), y(t)] = f[\lambda(t)]$ and $x'(t) + iy'(t) = \lambda'(t)$, which finally yields

$$\int_L f(z)\,dz = \int_\alpha^\beta f[\lambda(t)]\,\lambda'(t)\,dt.$$

This formula reduces the evaluation of a complex integral to that of an ordinary integral (of a complex function of a real variable t).

Example. Consider the integral $\int_{|z-a|=\rho} \frac{dz}{z-a}$, where the circle is traversed counterclockwise. The equation of the contour of the integration is $z = a + \rho e^{it}$ $(0 \leqslant t \leqslant 2\pi)$ and $\lambda'(t) = \frac{dz}{dt} = i\rho e^{it}$.

Hence,

$$\int\limits_{|z-a|=\rho} \frac{dz}{z-a} = \int\limits_0^{2\pi} \frac{1}{\rho e^{it}} i\rho e^{it}\, dt = i \int\limits_0^{2\pi} dt = 2\pi i,$$

which, of course, coincides with the result obtained in Sec. 5.1.

5.4
CAUCHY'S INTEGRAL THEOREM

In this section we will start the proof of a theorem that is central to the theory of analytic functions of a complex variable.

Cauchy's integral theorem (1825). *If $f(z)$ is a single-valued analytic function in a simply connected (in the finite plane) domain G, the equality*

$$\int\limits_L f(z)\, dz = 0$$

holds for every rectifiable closed curve L in G. We note that this theorem may easily be derived from the D'Alembert-Euler equations and Green's formula on the plane by the additional hypothesis that $f'(z)$ is continuous in G. Indeed, Green's formula,

$$\int\limits_L P\, dx + Q\, dy = \iint\limits_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\, dy\ ,$$

where D is the interior of the rectifiable Jordan curve L, is derived on the following assumptions:

(a) every straight line parallel to a coordinate axis intersects L not more than at two points (exceptions are allowed only for the two extreme points in the directions of the two axes, where intersection is possible along a rectilinear segment);

(b) P, Q, $\partial Q/\partial x$, and $\partial P/\partial y$ are continuous in the closed domain $\overline{D}$.

Assuming that L satisfies (a) and noting that from the continuity of $f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$ there follows the continuity of $\partial u/\partial x$, $\partial u/\partial y$, $\partial v/\partial x$, and $\partial v/\partial y$ (and, of course, the continuity of u and v), we find that

$$\int\limits_L f(z)\, dz = \int\limits_L (u\, dx - v\, dy) + i \int\limits_L (v\, dx + u\, dy)$$

$$= \iint\limits_D \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\, dy + i \iint\limits_D \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dx\, dy.$$

Due to the D'Alembert-Euler equations the expressions under the double-integral signs vanish, i.e.

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}, \quad \frac{\partial v}{\partial x}=-\frac{\partial u}{\partial y},$$

whence

$$\int_L f(z)\,dz=0.$$

Subsequently we will see that the derivation of an analytic function is always analytic and, hence, continuous. But this conclusion will be based on Cauchy's integral theorem. If we wish to escape a "vicious circle" in our reasoning, we must establish proof of this theorem without assuming the continuity of $f'(z)$. Such proof was first given by Edvard J. B. Goursat in 1900 and was later simplified by A. Pringsheim.

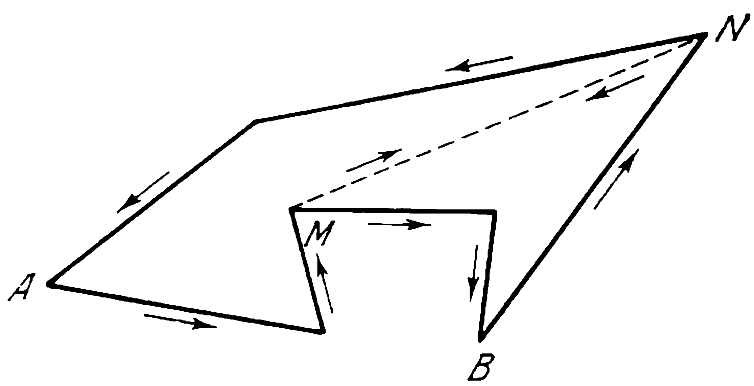

Fig. 33

We preface the proof by a lemma according to which the general case of a rectifiable curve is reduced to the simple case where L is the contour of a triangle.

Lemma. *If a function $f(z)$ is continuous in a simply connected domain G and if for any triangular contour Δ inside G we have $\int_\Delta f(z)\,dz=0$, then $\int_L f(z)\,dz=0$ for any closed curve L lying in G, too.*

Proof. Suppose at first that L is a closed polygon P. If the number of its vertices is $n>3$ and it does not have self-intersections, there exists a diagonal MN that belongs to the interior of the polygon (except points M and N that lie on P) and which dissects the interior of P into two polygons, $MNAM$ and $NMBN$ (Fig. 33) each of which has a number of sides smaller than n (prove this fact). Since

$$\int_{MNAM} f(z)\,dz=\int_{MN} f(z)\,dz+\int_{NAM} f(z)\,dz$$

and

$$\int_{NMBN} f(z)\,dz=\int_{NM} f(z)\,dz+\int_{MBN} f(z)\,dz,$$

we find that

$$\int_{MNAM} f(z)\,dz+\int_{NMBN} f(z)\,dz=\int_{NAM} f(z)\,dz+\int_{MBN} f(z)\,dz=\int_P f(z)\,dz.$$

If the lemma is true for a polygon with the number of vertices smaller than n, then it is valid for a polygon with n vertices. But by the hypothesis it is valid for a triangle. Therefore, by induction, it is true for any polygon without self-intersections. Next, let us consider a polygon P with n vertices that has self-intersections (Fig. 34). We traverse it in a given direction starting at point A_0 until a certain side meets for the first time a side traversed earlier. In our example this is side A_3A_4 intersecting side A_0A_1 at point B. Then the closed polygon $BA_1A_2A_3B$ does not have self-intersections and, hence, according to the above result

$$\int_P f(z)\,dz = \int_{BA_1A_2A_3B} f(z)\,dz + \int_{A_0BA_4A_5A_6A_7A_0} f(z)\,dz$$

$$= \int_{A_0BA_4A_5A_6A_7A_0} f(z)\,dz.$$

The number of vertices of the new polygon $A_0BA_4A_5A_6A_7$ is smaller than n because we introduced only one new vertex B (compared to P) and discarded together with $BA_1A_2A_3B$ at least two vertices of the initial polygon. It is necessary to note, however, that there is a possibility not provided for by the previous reasoning. Let us illustrate it with the example of polygon $A_0BA_4A_5A_6A_7$. If we traverse this polygon starting at A_0 in the direction given by the prescribed order of vertices, then when we move from A_4 to A_5 we meet points on the side BA_4 we traversed earlier, and not one of these self-intersection points is the first. In this case we must isolate the lune BA_4B, the integral along which is obviously zero. This yields

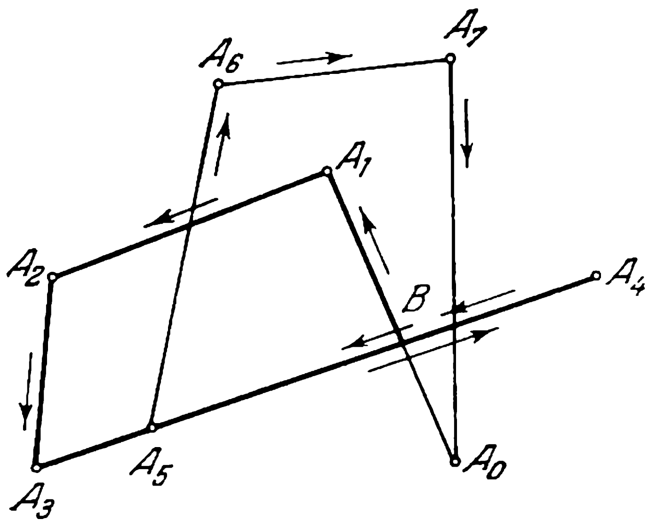

Fig. 34

$$\int_{A_0BA_4A_5A_6A_7A_0} f(z)\,dz = \int_{BA_4B} f(z)\,dz + \int_{A_0BA_5A_6A_7A_0} f(z)\,dz$$

$$= \int_{A_0BA_5A_6A_7A_0} f(z)\,dz.$$

The number of vertices of the last polygon is a unit less than the number of vertices of the previous one, since vertex B is discarded and no vertex is introduced instead.

Thus, we can always substitute for a polygon with self-intersections another with a smaller number of sides and yet not change the value of the integral. Consequently, here too we can complete the proof by induction, starting from a triangle. (For a lune, which is a straight segment traversed twice in the opposite directions, the proposition is obviously true.)

Suppose, finally, that L is an arbitrary closed rectifiable curve belonging to G. Consider a bounded closed subdomain E on G containing L and assume, for the sake of definiteness, that E is the set of all points whose distances from L is not greater than a definite $\delta > 0$ (δ must be smaller than the distance between L and the boundary of G).

If the function $f(z)$ is continuous on the bounded closed set E, it is uniformly continuous on E, and for every positive ε we will find a positive ρ such that $|f(\zeta_1) - f(\zeta_2)| < \varepsilon$ if $|\zeta_1 - \zeta_2| < \rho$ and $\zeta_j \in E$ $(j = 1, 2)$. Let us fix on L in a definite direction (the direction of the integration) the points $z_0, z_1, \ldots, z_n = z_0$ that partition it into arcs $l_0, l_1, \ldots, l_{n-1}$ in a way such that $\max\limits_{j=0, 1, \ldots, n-1}$ (length of l_j) $< \min(\delta, \rho)$. Connecting pairs of points z_j and z_{j+1} $(j = 0, 1, \ldots, n-1)$ by straight lines p_j, we obtain chords for L; since (length of p_j) $\leqslant$ (length of l_j) $< \delta$, all points of the p_j belong to E. The set of these chords constitute a closed broken line P inscribed in L and belonging to E. Since we have already proved that $\int\limits_P f(z)\, dz = 0$, we find that

$$\left|\int\limits_L f(z)\, dz\right| = \left|\int\limits_L f(z)\, dz - \int\limits_P f(z)\, dz\right|$$

$$= \left|\sum_{j=0}^{n-1} \int\limits_{l_j} f(z)\, dz - \sum_{j=0}^{n-1} \int\limits_{p_j} f(z)\, dz\right| \leqslant \sum_{j=0}^{n-1} \left|\int\limits_{l_j} f(z)\, dz - \int\limits_{p_j} f(z)\, dz\right|.$$

Noting that

$$\int\limits_{l_j} f(z_j)\, dz = f(z_j)(z_{j+1} - z_j) = \int\limits_{p_j} f(z_j)\, dz,$$

we obtain

$$\left|\int\limits_{l_j} f(z)\, dz - \int\limits_{p_j} f(z)\, dz\right| \leqslant \left|\int\limits_{l_j} [f(z) - f(z_j)]\, dz\right| + \left|\int\limits_{p_j} [f(z) - f(z_j)]\, dz\right|$$

$$\leqslant \sup_{z \in l_j} |f(z) - f(z_j)| \text{ (length of } l_j) + \sup_{z \in p_j} |f(z) - f(z_j)| \text{ (length of } p_j).$$

But any two points of the l_j or the p_j are separated by a distance less than ρ, whence

$$\sup_{z\in l_j} |f(z)-f(z_j)| < \varepsilon, \quad \sup_{z\in p_j} |f(z)-f(z_j)| < \varepsilon$$

and

$$\left|\int_{l_j} f(z)\,dz - \int_{p_j} f(z)\,dz\right| < \varepsilon \text{ (length of } l_j + \text{length of } p_j)$$

$$\leqslant 2\varepsilon \text{ (length of } l_j).$$

Hence

$$\left|\int_L f(z)\,dz\right| \leqslant 2\varepsilon \sum_{j=0}^{n-1} \text{(length of } l_j) = 2\varepsilon \text{ (length of } L),$$

which implies that $\int_L f(z)\,dz = 0$ because ε is arbitrarily small.

5.5

PROOF OF CAUCHY'S INTEGRAL THEOREM

Therefore, to prove Cauchy's integral theorem it suffices to prove it for the case where L is the contour Δ of a triangle belonging to domain G. Assume that

$$\left|\int_\Delta f(z)\,dz\right| = M \geqslant 0;$$

we wish to prove that $M = 0$. We decompose Δ into four triangles

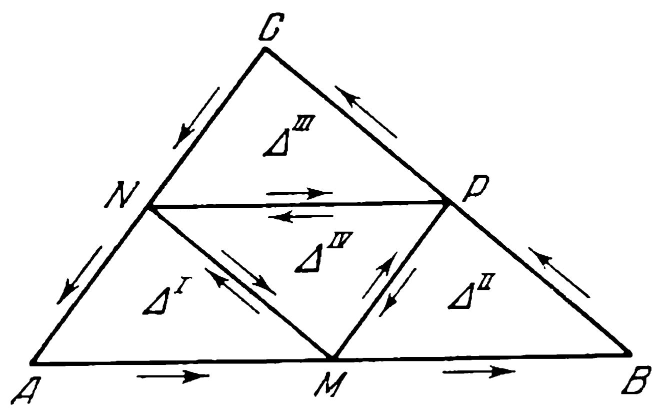

Fig. 35

Δ^{I}, Δ^{II}, Δ^{III}, and Δ^{IV} by connecting the midpoints of the sides of Δ (Fig. 35). Noting that

$$\int_{\Delta^{\mathrm{I}}} = \int_{NAM} + \int_{MN}, \quad \int_{\Delta^{\mathrm{II}}} = \int_{MBP} + \int_{PM},$$

$$\int_{\Delta^{\mathrm{III}}} = \int_{PCN} + \int_{NP}, \quad \int_{\Delta^{\mathrm{IV}}} = \int_{NM} + \int_{MP} + \int_{PN},$$

we obtain

$$\int_{\Delta^{\mathrm{I}}} + \int_{\Delta^{\mathrm{II}}} + \int_{\Delta^{\mathrm{III}}} + \int_{\Delta^{\mathrm{IV}}} = \int_{NAM} + \int_{MBP} + \int_{PCN} = \int_{\Delta},$$

whence

$$M = \left|\int_{\Delta}\right| \leqslant \left|\int_{\Delta^{\mathrm{I}}}\right| + \left|\int_{\Delta^{\mathrm{II}}}\right| + \left|\int_{\Delta^{\mathrm{III}}}\right| + \left|\int_{\Delta^{\mathrm{IV}}}\right|,$$

and, hence, at least one of the moduli $\left|\int_{\Delta^{\mathrm{I}}}\right|$, $\left|\int_{\Delta^{\mathrm{II}}}\right|$, $\left|\int_{\Delta^{\mathrm{III}}}\right|$, and $\left|\int_{\Delta^{\mathrm{IV}}}\right|$ is not smaller than $M/4$. We denote by Δ_1 the one of the triangles Δ^{I}, Δ^{II}, Δ^{III}, and Δ^{IV} for which

$$\left|\int_{\Delta_1} f(z)\,dz\right| \geqslant \frac{M}{4}.$$

Repeating for Δ_1 the line of reasoning followed for Δ, we find a triangle Δ_2 (one of the four triangles into which Δ_1 is decomposed by straight lines connecting the midpoints of Δ_1) such that

$$\left|\int_{\Delta_2} f(z)\,dz\right| \geqslant \frac{M}{4^2}.$$

In this way we arrive at a sequence of triangles Δ, Δ_1, Δ_2, ..., Δ_n, ..., each of which belongs to the previous one (we use the same symbol Δ_n to denote the contour of the triangle and the closed domain bounded by it; it is quite clear from the context which of the two meanings is used). Obviously,

$$\left|\int_{\Delta_n} f(z)\,dz\right| \geqslant \frac{M}{4^n}$$

and

$$\text{length of } \Delta_n = \frac{\text{length of } \Delta_{n-1}}{2} = \frac{\text{length of } \Delta}{2^n}.$$

Now we use the fact that $f(z)$ is analytic. All the triangles Δ_j ($\Delta_0 = \Delta$) have a common point ζ that belongs to the closed domain Δ and, hence, to domain G. Since $f(z)$ is differentiable at this point, for any positive ε there exists a positive such that

$$\left|\frac{f(z)-f(\zeta)}{z-\zeta} - f'(\zeta)\right| < \varepsilon \text{ for } |z-\zeta| < \delta.$$

We introduce the notation

$$\alpha(z, \zeta) = \frac{f(z)-f(\zeta)}{z-\zeta} - f'(\zeta).$$

Then

$$f(z) = f(\zeta) + f'(\zeta)(z - \zeta) + \alpha(z, \zeta)(z - \zeta),$$

where $|\alpha(z, \zeta)| < \varepsilon$ for $|z - \zeta| < \delta$. If n is sufficiently great ($n \geqslant n_0$), triangle Δ_n containing ζ belongs to the circle $|z - \zeta| < \delta$. Therefore, noting that $\int_{\Delta_n} dz = \int_{\Delta_n} z\,dz = 0$ (see Sec. 5.1), we find that

$$\left|\int_{\Delta n} f(z)\,dz\right| = \left|\int_{\Delta n} [f(\zeta) + f'(\zeta)(z-\zeta) + \alpha(z, \zeta)(z-\zeta)]\,dz\right|$$

$$= \left|f(\zeta)\int_{\Delta n} dz + f'(\zeta)\int_{\Delta n} z\,dz - \zeta f'(\zeta)\int_{\Delta n} dz + \int_{\Delta n} \alpha(z, \zeta)(z-\zeta)\,dz\right|$$

$$< \varepsilon \sup_{z \in \Delta n} |z-\zeta| \times (\text{length of } \Delta_n) < \varepsilon \times (\text{length of } \Delta_n)^2 = \varepsilon \frac{(\text{length of } \Delta)^2}{4^n}.$$

Comparing the upper and lower estimates for $\left|\int_{\Delta n} f(z)\,dz\right|$, we see that

$$\frac{M}{4^n} \leqslant \varepsilon \frac{(\text{length of } \Delta)^2}{4^n},$$

whence

$$M \leqslant \varepsilon \times (\text{length of } \Delta)^2.$$

But ε is arbitrarily small and, hence, $M = 0$, which completes the proof.

5.6

EVALUATION OF DEFINITE INTEGRALS

In his first works Cauchy used his theorem to evaluate various definite integrals of functions of a real variable (mainly improper integrals). To understand these applications of Cauchy's theorem, which produced this theorem, we present three examples.

1. The Fresnel integrals $\int_0^\infty \cos \xi^2\,d\xi$ and $\int_0^\infty \sin \xi^2\,d\xi$. To evaluate these integrals, which are often encountered in the theory of diffraction, we consider an auxiliary function of a complex variable, $F(z) = e^{iz^2}$. This function may be viewed as a composite function: $F(z) = \varphi[f(z)]$, where $f(z) = iz^2$ and $\varphi(\zeta) = e^{\zeta}$. By the rule of differentiation of a composite function, $F(z)$ is differentiable in the entire plane and $dF(z)/dz = 2ize^{iz^2}$. Therefore, Cauchy's integral theorem can be applied to $F(z)$.

Figure 36 depicts the curve L used as the path of the integration. It consists of the segment OA of the positive real axis of length R (a positive number), the arc AB of the circle of radius R with the center at the origin of coordinates, and the segment BO of the bisector of the angle of the first quadrant. The angle AOB is therefore equal to $\pi/4$. In view of the integral theorem,

$$\int_L e^{i\zeta^2}\,d\zeta=\int_{OA} e^{i\zeta^2}\,d\zeta+\int_{AB} e^{i\zeta^2}\,d\zeta+\int_{BO} e^{i\zeta^2}\,d\zeta=0.$$

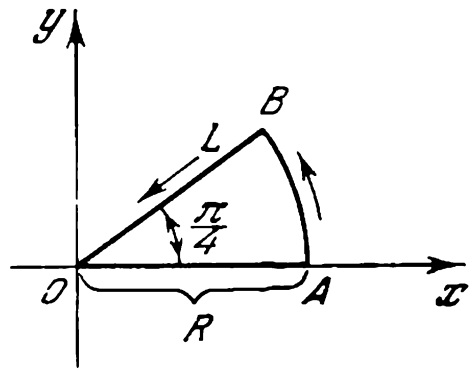

Fig. 36

On OA, the ζ is a real number ξ. Therefore, $d\zeta = d\xi$ and

$$J_1(R)=\int_{OA} e^{i\zeta^2}\,d\zeta=\int_0^R e^{i\xi^2}\,d\xi.$$

On AB the ζ is $R(\cos\varphi+i\sin\varphi)$, where φ varies from 0 to $\pi/4$. Therefore,

$$\zeta^2=R^2(\cos 2\varphi+i\sin 2\varphi)\quad\left(0\leqslant 2\varphi\leqslant\frac{\pi}{2}\right),$$

$$d\zeta=R(-\sin\varphi+i\cos\varphi)\,d\varphi=iR(\cos\varphi+i\sin\varphi)\,d\varphi$$

and

$$J_2(R)=\int_{AB} e^{i\zeta^2}\,d\zeta=\int_0^{\frac{\pi}{4}}\exp\left[iR^2(\cos 2\varphi+i\sin 2\varphi)\right]iR(\cos\varphi+i\sin\varphi)\,d\varphi.$$

Finally, on BO the ζ is $\rho\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)$, where ρ varies from R to 0. Therefore,

$$\zeta^2=\rho^2\left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right)=i\rho^2,\quad d\zeta=\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)d\rho$$

and

$$J_3(R)=\int_{BO} e^{i\zeta^2}\,d\zeta=\int_R^0 e^{-\rho^2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)d\rho$$

$$=-\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)\int_0^R e^{-\rho^2}\,d\rho=-\frac{\sqrt{2}}{2}(1+i)\int_0^R e^{-\rho^2}\,d\rho.$$

As $R\to\infty$, the integral $J_3(R)$ tends to the limit

$$-\frac{\sqrt{2}}{2}(1+i)\int_0^\infty e^{-\rho^2}\,d\rho=-\frac{\sqrt{2\pi}}{4}(1+i)$$

since

$$\int_0^{\infty} e^{-\rho^2}\,d\rho=\frac{\sqrt{\pi}}{2}.\,^*$$

Now we show that $J_2(R)\to 0$ as $R\to\infty$. To this end we estimate the modulus of $|J_2(R)|$. We have

$$|J_2(R)|\leqslant R\int_0^{\frac{\pi}{4}}|\exp[iR^2(\cos 2\varphi+i\sin 2\varphi)]|\,|\cos\varphi+i\sin\varphi|\,d\varphi.$$

Here

$$|\exp[iR^2(\cos 2\varphi+i\sin 2\varphi)]|$$
$$=\exp(-R^2\sin 2\varphi)\ \text{ and }\ |\cos\varphi+i\sin\varphi|=1;$$

---

* This can be proved in the following manner. Write

$$\int_{-R}^{R}\int_{-R}^{R}e^{-\xi^2-\eta^2}\,d\xi\,d\eta=\left(\int_{-R}^{R}e^{-\xi^2}\,d\xi\right)^2=4\left(\int_0^R e^{-\xi^2}\,d\xi\right)^2. \qquad (\alpha)$$

Inscribe a circle k in the square with its center at the origin of coordinates and its sides parallel to the coordinate axes (each side of the square is $2R$) and also circumscribe a circle K about the same square. Then in view of the fact that the integrand is positive we find that

$$\iint_k e^{-\xi^2-\eta^2}\,d\xi\,d\eta<\int_{-R}^{R}\int_{-R}^{R}e^{-\xi^2-\eta^2}\,d\xi\,d\eta<\iint_K e^{-\xi^2-\eta^2}\,d\xi\,d\eta,$$

or, after transformation from the Cartesian coordinates ξ and η to the polar coordinates ρ and φ and using (α), we find that

$$\int_0^{2\pi}\int_0^{R}e^{-\rho^2}\rho\,d\rho\,d\varphi<4\left(\int_0^R e^{-\xi^2}\,d\xi\right)^2<\int_0^{2\pi}\int_0^{R\sqrt{2}}e^{-\rho^2}\rho\,d\rho\,d\varphi.$$

Evaluating the integrals and extracting a square root of the terms in the inequality, we obtain

$$\sqrt{\pi(1-e^{-R^2})}<2\int_0^R e^{-\xi^2}\,d\xi<\sqrt{\pi(1-e^{-2R^2})},$$

whence

$$\lim_{R\to\infty}2\int_0^R e^{-\xi^2}\,d\xi=\sqrt{\pi},\ \text{ or }\ \int_0^{\infty}e^{-\xi^2}\,d\xi=\frac{\sqrt{\pi}}{2}.$$

whence

$$|J_2(R)| \leqslant R \int_0^{\frac{\pi}{4}} \exp(-R^2 \sin 2\varphi)\, d\varphi.$$

But $\sin 2\varphi \geqslant (2/\pi)\, 2\varphi$ for $0 \leqslant 2\varphi \leqslant \pi/2$.* Consequently,

$$|J_2(R)| \leqslant R \int_0^{\frac{\pi}{4}} \exp\left(-\frac{4}{\pi} R^2 \varphi\right) d\varphi$$

$$= R \left.\frac{\exp\left(-\frac{4}{\pi} R^2\varphi\right)}{-\frac{4}{\pi} R^2}\right|_0^{\frac{\pi}{4}} = \frac{\pi}{4}\, \frac{1 - e^{-R^2}}{R},$$

which yields

$$\lim_{R\to\infty} J_2(R) = 0.$$

Finally, let us consider

$$J_1(R) = \int_0^R e^{i\xi^2}\, d\xi = \int_0^R \cos \xi^2\, d\xi + i \int_0^R \sin \xi^2\, d\xi.$$

Since $J_1(R) + J_2(R) + J_3(R) = 0$ for any R, we see that $J_1(R) = -J_2(R) - J_3(R)$ and

$$\lim_{R\to\infty} J_1(R) = -\lim_{R\to\infty} J_2(R) - \lim_{R\to\infty} J_3(R) = \frac{\sqrt{2\pi}}{4}(1+i),$$

i.e.

$$\lim_{R\to\infty} \left(\int_0^R \cos \xi^2\, d\xi + i \int_0^R \sin \xi^2\, d\xi \right) = \frac{\sqrt{2\pi}}{4}(1+i).$$

---

* Indeed, the function $f(\alpha) = \sin \alpha/\alpha$ decreases in the interval $(0, \pi/2)$ since its derivative is

$$f'(\alpha) = \frac{\alpha \cos \alpha - \sin \alpha}{\alpha^2} = \frac{\cos \alpha\, (\alpha - \tan \alpha)}{\alpha^2} < 0$$

for $0 < \alpha < \pi/2$. Therefore, $f(\alpha) > f(\pi/2)$ if $\alpha < \pi/2$, i.e. $\sin \alpha/\alpha > 2/\pi$, or

$$\sin \alpha > \frac{2}{\pi}\alpha \quad \left(0 < \alpha < \frac{\pi}{2}\right).$$

This inequality becomes an equality at $\alpha = 0$ or $\alpha = \pi/2$.

From this it follows, first, that the integrals

$$\int_0^\infty \cos \xi^2 \, d\xi = \lim_{R\to\infty} \int_0^R \cos \xi^2 \, d\xi \quad \text{and} \quad \int_0^\infty \sin \xi^2 \, d\xi = \lim_{R\to\infty} \int_0^R \sin \xi^2 \, d\xi$$

exist and, second, that

$$\int_0^\infty \cos \xi^2 \, d\xi = \int_0^\infty \sin \xi^2 \, d\xi = \frac{\sqrt{2\pi}}{4}.$$

2. The integral

$$\int_{-\infty}^{+\infty} e^{-\lambda x^2} \cos (2\lambda \alpha x) \, dx \quad (\lambda > 0, \ \alpha > 0).$$

To evaluate this integral, we integrate the function $f(z) = e^{-\lambda z^2}$ along the contour L of the rectangle depicted in Fig. 37. Since this

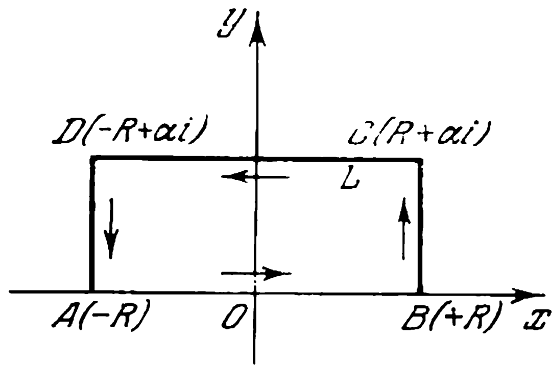

Fig. 37

function is differentiable in the entire plane, we apply Cauchy's integral theorem. This yields

$$\int_L e^{-\lambda\zeta^2} \, d\zeta = \int_{AB} e^{-\lambda\zeta^2} \, d\zeta + \int_{BC} e^{-\lambda\zeta^2} \, d\zeta + \int_{CD} e^{-\lambda\zeta^2} \, d\zeta + \int_{DA} e^{-\lambda\zeta^2} \, d\zeta = 0.$$

On AB,

$$\zeta = x \quad (-R \leqslant x \leqslant R) \text{ and } d\zeta = dx,$$

whence

$$J_1 = \int_{AB} e^{-\lambda\zeta^2} \, d\zeta = \int_{-R}^{+R} e^{-\lambda x^2} \, dx.$$

On BC,

$$\zeta = R + iy \quad (0 \leqslant y \leqslant \alpha),$$
$$\zeta^2 = R^2 + 2iRy - y^2 \text{ and } d\zeta = i \, dy,$$

whence

$$J_2=\int_{BC} e^{-\lambda\zeta^2}\,d\zeta=\int_0^{\alpha}\exp\left[-\lambda\left(R^2+i2Ry-y^2\right)\right] i\,dy$$

$$=i\int_0^{\alpha}\exp\left[-\lambda\left(R^2-y^2\right)\right]\exp\left(-2Riy\lambda\right)dy.$$

On CD,

$\zeta=x+i\alpha \quad (R\geqslant x\geqslant -R)$, $\zeta^2=x^2+2i\alpha x-\alpha^2$, and $d\zeta=dx$,

whence

$$J_3=\int_{CD} e^{-\lambda\zeta^2}\,d\zeta=\int_R^{-R}\exp\left[-\lambda\left(x^2+2i\alpha x-\alpha^2\right)\right]dx$$

$$=-e^{\lambda\alpha^2}\int_{-R}^{+R}\exp\left(-\lambda x^2-2i\alpha\lambda x\right)dx$$

$$=-e^{\lambda\alpha^2}\int_{-R}^{+R}e^{-\lambda x^2}\left[\cos\left(2\lambda\alpha x\right)-i\sin\left(2\lambda\alpha x\right)\right]dx.$$

Finally, on DA,

$\zeta=-R+iy \quad (\alpha\geqslant y\geqslant 0)$, $\zeta^2=R^2-2Riy-y^2$, and $d\zeta=i\,dy$,

whence

$$J_4=\int_{DA} e^{-\lambda\zeta^2}\,d\zeta=\int_{\alpha}^{0}\exp\left[-\lambda\left(R^2-2Riy-y^2\right)\right] i\,dy$$

$$=-i\int_0^{\alpha}\exp\left[-\lambda\left(R^2-y^2\right)\right]\exp\left(2Riy\lambda\right)dy.$$

Now let us make R increase without limit. Then the integrals J_2 and J_4 tend to zero. Indeed,

$$|J_2|\leqslant\int_0^{\alpha}\exp\left[-\lambda\left(R^2-y^2\right)\right]\left|\exp\left(-2iR\lambda y\right)\right|dy=\int_0^{\alpha}e^{-\lambda(R^2-y^2)}\,dy.$$

For $R>\alpha$ we find that

$$|J_2|\leqslant\int_0^{\alpha}\exp\left[-\lambda\left(R^2-\alpha^2\right)\right]dy$$

$$=\alpha\exp\left[-\lambda\left(R^2-\alpha^2\right)\right]\to 0 \quad (R\to\infty).$$

Similarly,

$$|J_4| \leqslant \int_0^{\alpha} \exp[-\lambda(R^2-y^2)] \, |\exp 2iR\lambda y| \, dy$$

$$= \int_0^{\alpha} \exp[-\lambda(R^2-y^2)] \, dy \to 0 \quad (R\to\infty).$$

As $R\to\infty$ the integral J_1 tends to

$$\int_{-\infty}^{+\infty} \exp(-\lambda x^2) \, dx = \frac{1}{\sqrt{\lambda}} \int_{-\infty}^{+\infty} \exp[-(\sqrt{\lambda}x)^2] \, d(\sqrt{\lambda}x) = \sqrt{\frac{\pi}{\lambda}}$$

$\left(\text{since } \int_{-\infty}^{+\infty} e^{-x^2} \, dx = \sqrt{\pi}; \text{ see the previous example}\right)$.

Finally, from the fact that

$$J_1 + J_2 + J_3 + J_4 = 0$$

we conclude that

$$e^{\lambda\alpha^2} \int_{-\infty}^{+\infty} e^{-\lambda x^2} [\cos(2\lambda\alpha x) - i\sin(2\lambda\alpha x)] \, dx$$

$$= -\lim_{R\to\infty} J_3 = \lim_{R\to\infty} J_1 + \lim_{R\to\infty} J_2 + \lim_{R\to\infty} J_4 = \sqrt{\frac{\pi}{\lambda}}.$$

Comparing the real parts in this relationship, we obtain

$$e^{\lambda\alpha^2} \int_{-\infty}^{+\infty} e^{-\lambda x^2} \cos(2\lambda\alpha x) \, dx = \sqrt{\frac{\pi}{\lambda}},$$

i. e.

$$\int_{-\infty}^{+\infty} e^{-\lambda x^2} \cos(2\lambda\alpha x) \, dx = \sqrt{\frac{\pi}{\lambda}} \, e^{-\lambda\alpha^2}.$$

3. The integral $\int_0^{\infty} \frac{\sin\xi}{\xi} \, d\xi$. We take an auxiliary function $f(z) = e^{iz}/z$. This function is defined on a domain G that is the entire plane without the origin of coordinates and is differentiable in this domain. We take the contour of the integration L as depicted in Fig. 38 (this contour and all the points of its interior lie in a simply connected domain D bounded by the negative half of the imaginary axis). If we apply the integral theorem to $f(z) = e^{iz}/z$, we find:

$$\int_L \frac{e^{i\zeta} \, d\zeta}{\zeta} = \int_{AB} \frac{e^{i\zeta} \, d\zeta}{\zeta} + \int_{BCD} \frac{e^{i\zeta} \, d\zeta}{\zeta} + \int_{DE} \frac{e^{i\zeta} \, d\zeta}{\zeta} + \int_{EFA} \frac{e^{i\zeta} \, d\zeta}{\zeta} = 0.$$

On AB, the ζ is equal to a real number ξ. Therefore, $d\zeta = d\xi$ and

$$J_1 = \int_{AB} \frac{e^{i\zeta}\,d\zeta}{\zeta} = \int_r^R \frac{e^{i\xi}\,d\xi}{\xi} = \int_r^R \frac{\cos\xi\,d\xi}{\xi} + i\int_r^R \frac{\sin\xi\,d\xi}{\xi}.$$

On BCD, the ζ is equal to $R(\cos\varphi + i\sin\varphi)$ $(0 \leqslant \varphi \leqslant \pi)$; whence

$$d\zeta = R(-\sin\varphi + i\cos\varphi)\,d\varphi = iR(\cos\varphi + i\sin\varphi)\,d\varphi$$

and

$$J_2 = \int_{BCD} \frac{e^{i\zeta}\,d\zeta}{\zeta} = \int_0^\pi \frac{\exp[iR(\cos\varphi + i\sin\varphi)]\,iR(\cos\varphi + i\sin\varphi)\,d\varphi}{R(\cos\varphi + i\sin\varphi)}$$

$$= i\int_0^\pi \exp(iR\cos\varphi - R\sin\varphi)\,d\varphi.$$

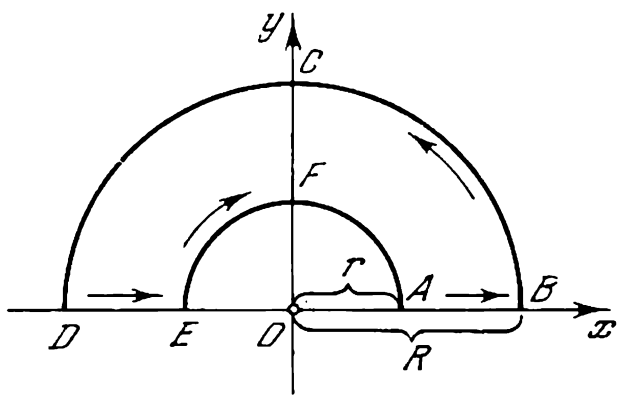

Fig. 38

On DE, the ζ is equal to the real number ξ, whence $d\zeta = d\xi$ and

$$J_3 = \int_{DE} \frac{e^{i\zeta}\,d\zeta}{\zeta} = \int_{-R}^{-r} \frac{e^{i\xi}\,d\xi}{\xi} = \int_{-R}^{-r} \frac{\cos\xi\,d\xi}{\xi} + i\int_{-R}^{-r} \frac{\sin\xi\,d\xi}{\xi}.$$

Changing the integration variable from ξ to $-\xi$, we find that

$$J_3 = -\int_r^R \frac{\cos\xi\,d\xi}{\xi} + i\int_r^R \frac{\sin\xi\,d\xi}{\xi}.$$

Finally, on EFA,

$$\zeta = r(\cos\varphi + i\sin\varphi) \quad (\pi \geqslant \varphi \geqslant 0),$$

whence

$$d\zeta = ir(\cos\varphi + i\sin\varphi)\,d\varphi$$

and

$$J_4=\int\limits_{EFA}\frac{e^{i\zeta}\,d\zeta}{\zeta}=\int\limits_{\pi}^{0}\frac{\exp\left[ir(\cos\varphi+i\sin\varphi)\right]ir(\cos\varphi+i\sin\varphi)}{r(\cos\varphi+i\sin\varphi)}\,d\varphi$$

$$=-i\int\limits_{0}^{\pi}\exp\left(ir\cos\varphi-r\sin\varphi\right)d\varphi .$$

Now we make R grow without limit; the integral

$$J_2=i\int\limits_{0}^{\pi}\exp\left(iR\cos\varphi-R\sin\varphi\right)d\varphi$$

then tends to zero. Indeed,

$$|J_2|=\left|\int\limits_{0}^{\pi}\exp\left(iR\cos\varphi-R\sin\varphi\right)d\varphi\right|\leqslant\int\limits_{0}^{\pi}\exp\left(-R\sin\varphi\right)d\varphi$$

$$=2\int\limits_{0}^{\pi/2}\exp\left(-R\sin\varphi\right)d\varphi<2\int\limits_{0}^{\pi/2}\exp\left(-R\frac{2}{\pi}\varphi\right)d\varphi$$

$$=\pi\frac{1-e^{-R}}{R}<\frac{\pi}{R},$$

from which it follows that $\lim\limits_{R\to\infty}J_2=0$.

Finally, suppose r tends to zero. Let us find the limit for

$$J_4=-i\int\limits_{0}^{\pi}\exp\left(ir\cos\varphi-r\sin\varphi\right)d\varphi.$$

Since $\exp(iz)$ is continuous at point $z=0$, where it is equal to unity, for any positive ε we can indicate a $\delta(\varepsilon)>0$ such that for $|z|=r<\delta(\varepsilon)$ the inequality

$$|\exp(iz)-1|=|\exp(ir\cos\varphi-r\sin\varphi)-1|<\varepsilon$$

is valid. This implies that

$$\left|J_4-\left(-i\int\limits_{0}^{\pi}1\times d\varphi\right)\right|=\left|\int\limits_{0}^{\pi}\left[\exp\left(ir\cos\varphi-r\sin\varphi\right)-1\right]d\varphi\right|<\pi\varepsilon,$$

i.e.

$$\lim_{r\to 0}J_4=-i\int\limits_{0}^{\pi}1\times d\varphi=-\pi i.$$

Returning to the main relationship

$$\int\limits_{L}\frac{e^{i\zeta}\,d\zeta}{\zeta}=J_1+J_2+J_3+J_4=0,$$

we see that $J_1 + J_3 = -J_2 - J_4$ or, using the above expressions for J_1 and J_3,

$$2i \int_0^R \frac{\sin \xi \, d\xi}{\xi} = -J_2 - J_4.$$

As $R \to \infty$ and $r \to 0$, the right-hand side, as we have seen, tends to the limit $-\lim\limits_{R\to\infty} J_2 - \lim\limits_{r\to 0} J_4 = \pi i$. Therefore, the left-hand side also tends to that limit:

$$\lim_{\substack{R\to\infty \\ r\to 0}} 2i \int_r^R \frac{\sin \xi}{\xi} \, d\xi = \pi i.$$

Since we have proved that this limit exists, we may denote it by $\int\limits_0^\infty \frac{\sin \xi}{\xi} \, d\xi$ and write

$$\int_0^\infty \frac{\sin \xi}{\xi} \, d\xi = \frac{\pi}{2}.$$

5.7

THE INTEGRAL AND THE PRIMITIVE

From Cauchy's integral theorem it follows that integrals of analytic functions along any two curves L_1 and L_2 with the same lower and upper ends z_0 and z (Fig. 39) have the same value. Indeed, the curve $L = L_1 + (-L_2)$ is closed and, therefore,

$$\int_L f(z)\, dz = \int_{L_1} f(z)\, dz + \int_{-L_2} f(z)\, dz$$
$$= \int_{L_1} f(z)\, dz - \int_{L_2} f(z)\, dz = 0,$$

whence

$$\int_{L_1} f(z)\, dz = \int_{L_2} f(z)\, dz.$$

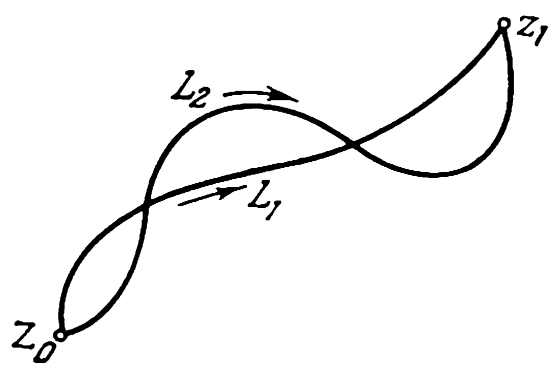

Fig. 39

Hence, *in a simply connected domain G, integrals of a function f (z) that is analytic in G depend only on the upper and lower ends of the path of integration. Consequently, for the integral along an arbitrary rectifiable curve L connecting points z_0 and z we can use the notation* $\int\limits_{z_0}^{z} f(z)\, dz$. We will call z_0 and z the *lower and upper limits of integration.*

If we fix z_0, we can consider $\int\limits_{z_0}^{z} f(z)\,dz$ as a function of the upper limit of integration:

$$\int\limits_{z_0}^{z} f(z)\,dz = F(z).$$

Let us show that $F(z)$ *is an analytic function and that its derivative is* $f(z)$. For future purposes we give a more general

Theorem. *Suppose that $f(z)$ is a function continuous in G and that the integrals of this function along every rectifiable curve in G depend only on the upper and lower ends of the curve. Then the integral*

$$F(z) = \int\limits_{z_0}^{z} f(z)\,dz$$

is an analytic function and

$$F'(z) = f(z).$$

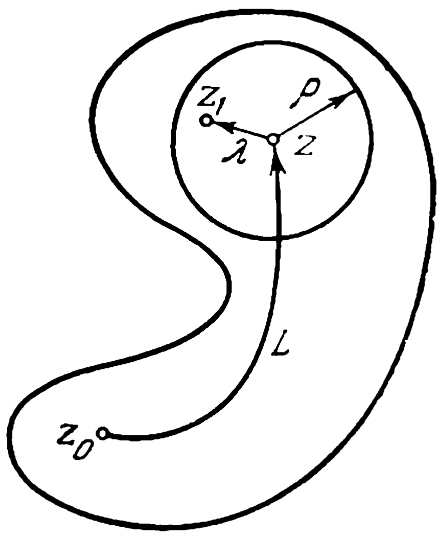

Fig. 40

Proof. Consider a neighborhood U of point z, e.g. $|\zeta - z| < \rho$, that belongs to G. We assume that $L \subset G$ is a curve connecting z_0 with z and λ is a straight segment connecting z with another point $z_1 \neq z$, $z_1 \in U$ (Fig. 40). Then

$$F(z) = \int\limits_{L} f(z)\,dz, \quad F(z_1) = \int\limits_{L+\lambda} f(z)\,dz = \int\limits_{L} f(z)\,dz + \int\limits_{\lambda} f(z)\,dz$$

and, hence,

$$F(z_1) - F(z) = \int\limits_{\lambda} f(\zeta)\,d\zeta,$$

$$\frac{F(z_1)-F(z)}{z_1-z} - f(z) = \frac{\int\limits_{\lambda} f(\zeta)\,d\zeta - f(z)(z_1-z)}{z_1-z} = \frac{\int\limits_{\lambda} [f(\zeta)-f(z)]\,d\zeta}{z_1-z}.$$

In view of the continuity of $f(z)$, for any positive ε we can choose the radius ρ of a neighborhood U so small that for any point $\zeta \in U$ the inequality $|f(\zeta) - f(z)| < \varepsilon$ is valid. Then we have

$$\left|\frac{F(z_1)-F(z)}{z_1-z} - f(z)\right| < \frac{\varepsilon \times (\text{length of } \lambda)}{|z_1-z|} = \varepsilon$$

(since $|z_1 - z| = (\text{length of } \lambda)$); whence

$$\lim_{z_1 \to z} \frac{F(z_1)-F(z)}{z_1-z} = f(z) = F'(z).$$

We call a function $\Phi(z)$ that is analytic in a domain G the ***primitive of*** $f(z)$ if $\Phi'(z) = f(z)$ at all points of G. Going by the above theorem, we can say that $\int_{z_0}^{z} f(z)\,dz$ is a primitive of $f(z)$.

Let us show that all primitives of $f(z)$ are expressed by the formula

$$\Phi(z) = \int_{z_0}^{z} f(z)\,dz + C, \tag{5.1}$$

where C is constant. We put

$$\omega(z) = \Phi(z) - \int_{z_0}^{z} f(z)\,dz = u(x, y) + iv(x, y).$$

Then

$$\omega'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y} = f(z) - f(z) \equiv 0,$$

whence

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} = \frac{\partial u}{\partial y} \equiv 0$$

and, therefore,

$$u(x, y) = C_1, \quad v(x, y) = C_2,$$

i.e.

$$\omega(z) = u(x, y) + iv(x, y) \equiv C_1 + iC_2 = C.$$

Putting $z = z_0$ in (5.1), we find that

$$\Phi(z_0) = C,$$

whence

$$\int_{z_0}^{z} f(z)\,dz = \Phi(z) - \Phi(z_0). \tag{5.2}$$

This result enables us to reduce the evaluation of an integral of an analytic function $f(z)$ to finding a primitive of the function. For instance, if we note that in the domain $-\pi < \arg z < \pi$ the function $\ln z = \ln|z| + i\arg z$ is a primitive of $f(z) = 1/z$, we can at once write

$$\int_{1}^{z} \frac{dz}{z} = \ln z - \ln 1 = \ln z.$$

5.8

GENERALIZATION OF CAUCHY'S INTEGRAL THEOREM TO FUNCTIONS NONANALYTIC ON THE INTEGRATION CONTOUR

Cauchy's integral theorem can be generalized in various directions. *Suppose that G is the interior of a rectifiable Jordan curve Γ and $f(z)$ is a function continuous in the closed domain $\overline{G}$ and analytic in G. Then $\int_\Gamma f(z)\,dz = 0$.* The generalization here consists in the fact that $f(z)$ is not assumed to be analytic on the contour of the integration. We prove the above statement by the additional assumptions that each ray emerging from a point $z_0 \in G$ interesects L at one point only and that L consists of a finite number of smooth arcs. These conditions are obviously met if L is a convex polygon (e.g. a triangle) or a circle.

Let $z = z_0 + \lambda(t)$ $(0 \leqslant t \leqslant 2\pi)$ be the equation of L, where $\lambda(t)$ has, by our assumption, a piecewise continuous derivative $\lambda'(t)$. We subject L to a similarity transformation with respect to ρ $(0 < \rho < 1)$ with the center at z_0; we obtain a closed curve $L_\rho: z = z_0 + \rho\lambda(t)$ $(0 \leqslant t \leqslant 2\pi)$ in G. By Cauchy's integral theorem,

$$\int_{L_\rho} f(z)\,dz = \int_0^{2\pi} f[z_0 + \rho\lambda(t)]\,\rho\lambda'(t)\,dt = 0,$$

whence

$$\int_0^{2\pi} f[z_0 + \rho\lambda(t)]\,\lambda'(t)\,dt = 0.$$

Therefore,

$$\left|\int_L f(z)\,dz\right| = \left|\int_0^{2\pi} f[z_0 + \lambda(t)]\,\lambda'(t)\,dt\right|$$

$$= \left|\int_0^{2\pi} \{f[z_0 + \lambda(t)] - f[z_0 + \rho\lambda(t)]\}\,\lambda'(t)\,dt\right|$$

$$\leqslant \int_0^{2\pi} |f[z_0 + \lambda(t)] - f[z_0 + \rho\lambda(t)]|\,|\lambda'(t)|\,dt.$$

But $f(z)$ is uniformly continuous in $\overline{G}$ and, therefore, for any positive ε,

$$|f(z') - f(z'')| < \varepsilon \text{ if } |z' - z''| < \delta(\varepsilon),\ z',\ z'' \in \overline{G}.$$

We denote the upper bounds of $\lambda(t)$ and $\lambda'(t)$ in the interval $[0, 2\pi]$ by λ and λ', respectively. Then

$$|[z_0 + \lambda(t)] - [z_0 + \rho\lambda(t)]| \leqslant (1 - \rho)\lambda < \delta(\varepsilon)$$

if $1 - \rho < \delta(\varepsilon)/\lambda$ and, hence,

$$\left|\int_L f_1(z)\, dz\right| < \varepsilon\lambda' 2\pi.$$

Since ε is arbitrarily small, it follows that

$$\int_L f(z)\, dz = 0.$$

5.9 THE COMPOSITE CONTOUR THEOREM

Now suppose that $f(z)$ is a function analytic in a domain G that is not simply connected in the finite plane. If L is a closed rectifiable curve in G, its interior may either belong to G or not belong to G. In the first instance, we can apply to $\int_L f(z)\, dz$ the same line of reasoning which we followed in proving Cauchy's integral theorem, in which case we again find that $\int_L f(z)\, dz = 0$. In the second case we cannot use the line of reasoning up to the end.

Indeed, we can no longer state that the interior of any polygon inscribed in L and the interior of any of the triangles into which the polygon is decomposed belong to G. Consequently, we cannot rely on the fact that there exists a derivative of $f(z)$ at a point ζ in such a triangle (compare with Sec. 5.5).

A simple example is the function $f(z) = 1/z$, which is analytic in a domain G that is the finite complex plane without the origin of coordinates (this domain is not simply connected in the finite plane). In such a domain there exists closed rectifiable curves L such that $\int_L f(z)\, dz \neq 0$. Every each circle with its center at origin of coordinates will be such a curve because $\int_{|z|=\rho} dz/z = 2\pi i \neq 0$ (see Example (c) in Sec. 5.1).

Now let us show that with Cauchy's integral theorem we can prove that in the general case of a function $f(z)$ analytic in an arbitrary domain G (generally multiply connected) the evaluation of integrals of this function along a closed curve can be reduced to evaluating integrals along closed curves lying in the interior of the given curve. More exactly, we *assume that Γ is a rectifiable Jordan curve lying*

inside G and $\gamma_1, \gamma_2, \ldots, \gamma_n$ are rectifiable Jordan curves lying in the interior of Γ. In addition, we require that each closed curve γ_j lie in the exterior of any other curve γ_k ($j, k = 1, 2, \ldots, n$; $j \neq k$) and, finally, that the multiply connected domain D bounded by the curves Γ, $\gamma_1, \gamma_2, \ldots, \gamma_n$ lie in G (Fig. 41). *Under these conditions*

$$\int_{\Gamma} f(z)\,dz = \int_{\gamma_1} f(z)\,dz + \ldots + \int_{\gamma_n} f(z)\,dz, \tag{5.3}$$

where all the paths of integration are traversed in the same direction, e.g. counterclockwise.

To prove the above statement, we connect Γ with γ_1, γ_1 with $\gamma_2, \ldots, \gamma_{n-1}$ with γ_n, and γ_n with Γ by rectifiable Jordan arcs' e.g. broken lines, in a way such that these arcs, $\delta_1, \delta_2, \ldots, \delta_{n+1}$, pairwise, have no common points and belong to D except the ends

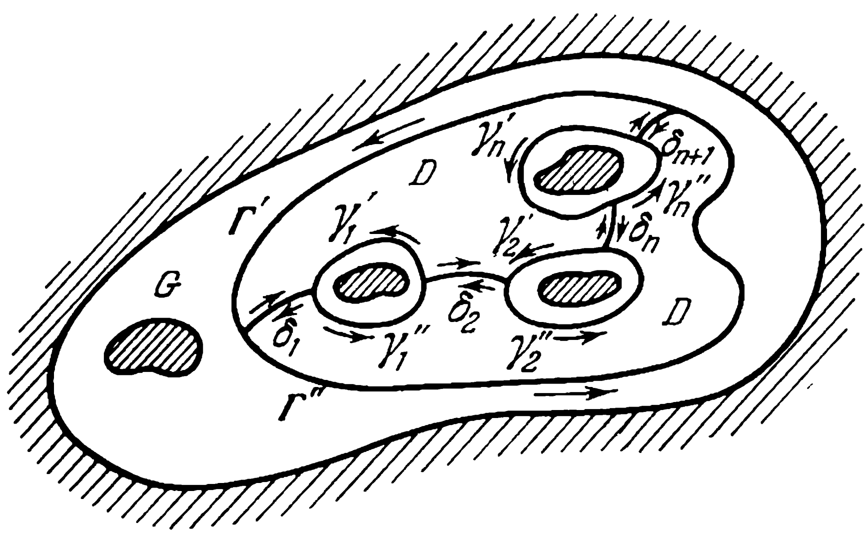

Fig. 41

of arcs that are on the boundary of D. The lower and upper points of $\delta_1, \delta_2, \ldots, \delta_{n+1}$ decompose each curve $\Gamma, \gamma_1, \gamma_2, \ldots, \gamma_n$ into two arcs, which we denote by the same symbol as the respective curve but add one or two primes. These arcs can be placed in such an order that the interior of each of the two Jordan curves composed of the arcs Γ', δ_1, γ_1', δ_2, γ_2', $\ldots$, γ_n', δ_{n+1} and Γ'', δ_{n+1}, γ_n'', δ_n, $\ldots$, γ_1'', δ_1, respectively, belongs to D and, hence, to G. Integrating $f(z)$ along these closed curves in the direction that we agreed upon earlier in relation to Γ (see Fig. 41) and noting that γ_k' and γ_k'' are traversed in the direction opposite to the one chosen for γ_k ($k = 1, 2, \ldots, n$) we have

$$\int\limits_{\Gamma'+\delta_1-\gamma_1'+\delta_2-\gamma_2'+\ldots-\gamma_n'+\delta_{n+1}} f(z)\,dz$$

$$= \int_{\Gamma'} f(z)\,dz + \int_{\delta_1} f(z)\,dz - \int_{\gamma_1'} f(z)\,dz + \int_{\delta_2} f(z)\,dz$$

$$-\int_{\gamma_2'} f(z)\,dz + \ldots - \int_{\gamma_n'} f(z)\,dz + \int_{\delta_{n+1}} f(z)\,dz = 0,$$

$$\int_{\Gamma''-\delta_{n+1}-\gamma_n''-\ldots-\delta_2-\gamma_1''-\delta_1} f(z)\,dz$$

$$= \int_{\Gamma''} f(z)\,dz - \int_{\delta_{n+1}} f(z)\,dz - \int_{\gamma_n''} f(z)\,dz -$$

$$\ldots - \int_{\delta_2} f(z)\,dz - \int_{\gamma_1''} f(z)\,dz - \int_{\delta_1} f(z)\,dz = 0.$$

We add these relations termwise. The integrals along the arcs $\delta_1, \delta_2, \ldots, \delta_{n+1}$ pairwise cancel out, and the integrals along the arcs Γ' and Γ'', γ_1' and γ_1'', $\ldots$, γ_n' and γ_n'' pairwise give integrals along the corresponding closed curves. Consequently,

$$\int_{\Gamma} f(z)\,dz - \int_{\gamma_1} f(z)\,dz - \ldots - \int_{\gamma_n} f(z)\,dz = 0, \tag{5.3'}$$

which we set out to prove.

Let us return to the function $f(z) = 1/z$ in the domain $G: z \neq 0$. If Γ is a rectifiable Jordan curve containing in its interior the origin of coordinates, then by the above theorem we have

$$\int_{\Gamma} \frac{1}{z}\,dz = \int_{\gamma} \frac{1}{z}\,dz = 2\pi i.$$

The above theorem can be stated somewhat differently. Let us think of the boundary of the multiply connected domain D as consisting of separate curves $\Gamma, \gamma_1, \gamma_2, \ldots, \gamma_n$ as a single *composite contour* L. The *positive direction* on L is chosen in a way such that the outer contour Γ is traversed counterclockwise while the inner contours are traversed clockwise. In other words, when we traverse the composite contour L in the positive direction, the interior D of the contour must always be to the left of us. We can now write the expression on the left-hand side of (5.3′) as $\int_L f(z)\,dz$ and state that *the integral of $f(z)$ along a composite contour L belonging to G together with its interior D is zero.*

This theorem, obviously, is a generalization of Cauchy's integral theorem to the case of composite contours in multiply connected domains. For the sake of convenience we will refer to the theorem proved in this section as the *composite contour theorem.*

5.10
INTEGRALS IN MULTIPLY CONNECTED DOMAINS

The composite contour theorem enables studying integrals of analytic functions in multiply connected domains. Suppose that G is multiply connected and $f(z)$ is single-valued and analytic in G. If $\int_L f(z)\,dz = 0$ for each closed rectifiable curve in G (this is the case with $f(z) = 1/(1+z^2)$ in the annulus $1 < |z| < R$, where R is any real number greater than unity), then to $\int_{z_0}^{z} f(z)\,dz$ we can apply the results obtained in Sec. 5.7. We will find that this integral is a single-valued analytic function, or a primitive of $f(z)$.

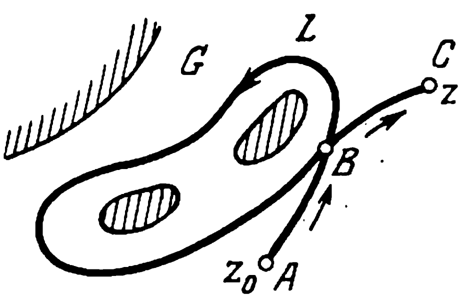

Fig. 42

Now suppose that in G there are such closed paths L (at least one) for which $\int_L f(z)\,dz \neq 0$ (this is the case with the function $1/z$ in the domain $z \neq 0$ or with the function $1/(1+z^2)$ in the domain $1+z^2 \neq 0$, i.e. $z \neq \pm i$; here $\int_{|z|=\rho} \frac{1}{z}\,dz = 2\pi i$ and, for $r < 2$, $\int_{|z-i|=r} \frac{dz}{1+z^2} = \pi$, and $\int_{|z+i|=r} \frac{dz}{1+z^2} = -\pi$).

We shall show that then for any two points z_0 and z in G there exist at least two paths (in Fig. 42 these are $L_1 = AB + BC$ and $L_2 = AB + L + BC$) connecting points $A(z_0)$ with $C(z)$ along which the integrals of $f(z)$ have different values. Indeed,

$$\int_{L_2} f(z)\,dz - \int_{L_1} f(z)\,dz = \left(\int_{AB} + \int_{L} + \int_{BC}\right) - \left(\int_{AB} + \int_{BC}\right) = \int_{L} f(z)\,dz \neq 0.$$

Therefore, the function $F(z) = \int_{z_0}^{z} f(z)\,dz$ is multiple-valued in G.

Let us use the example of $f(z) = 1/z$ to examine the nature of this multivaluedness in $G: z \neq 0$. We will see that the integral $\int_L (1/z)\,dz$, where $L \subset G$ is an arbitrary rectifiable curve connecting points $z_0 = 1$ and z, can be written in the form of the sum of the integral along any other curve l connecting the same two points and an integral multiple of the integral along a Jordan curve γ that contains point $z = 0$ in its interior (note that point $z = 0$ does not be-

long to G). Figure 43 presents L, l, and γ. To L we adjoin the arcs AB and AC, which in the course of integration are traversed twice but in the opposite directions. We have

$$\int_L \frac{dz}{z} = \int_{AaBA} \frac{dz}{z} + \int_{ABbCA} \frac{dz}{z} + \int_{ACD} \frac{dz}{z}.$$

Each of the two curves along which the first two integrals on the right-hand side are evaluated and the closed curve γ form a composite contour. Consequently, by the theorem of Sec. 5.9,

$$\int_{AaBA} \frac{dz}{z} = \int_\gamma \frac{dz}{z}, \quad \int_{ABbCA} \frac{dz}{z} = \int_\gamma \frac{dz}{z}.$$

Moreover, the curves l and DCA form a closed contour whose interior belongs entirely to G and, therefore, the integral along this contour is equal to zero. Whence

$$\int_{ACD} \frac{dz}{z} = \int_l \frac{dz}{z}.$$

Hence

$$\int_L \frac{dz}{z} = \int_l \frac{dz}{z} + 2\int_\gamma \frac{dz}{z}.$$

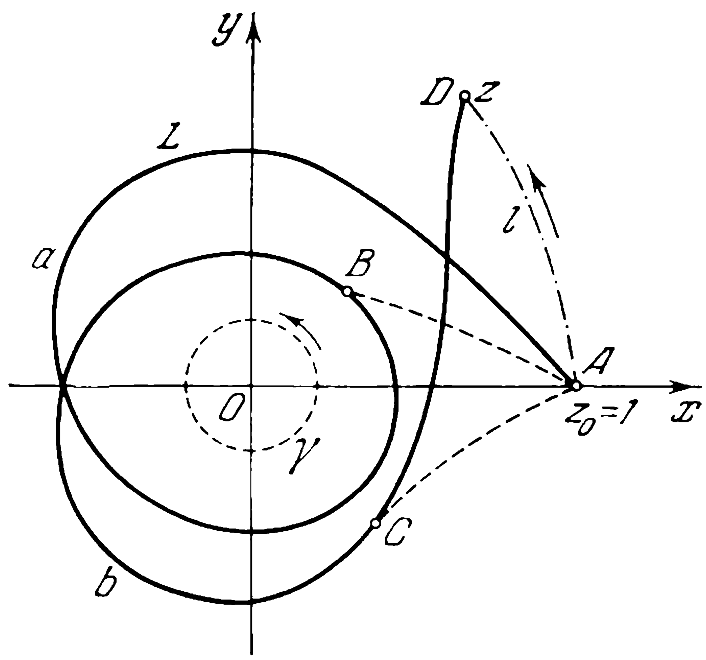

Fig. 43

This reasoning is of a general nature: for an arbitrary curve L we have

$$\int_L \frac{dz}{z} = \int_l \frac{dz}{z} + n\int_\gamma \frac{dz}{z},$$

where n is an integer that may be zero or even negative (in the last case the closed contours are traversed in the negative direction). Note that one can always choose a curve L for a given value of n.

The closed curve γ may be a circle $|z| = \rho$; then $\int_\gamma \frac{dz}{z} = 2\pi i$. Further, if point z does not lie on the negative half of the real axis ($|\arg z| < \pi$), then l can be a curve lying in the domain $|\arg z| < \pi$. We find then that $\int_l \frac{dz}{z} = \ln z$ (see Sec. 5.7). But if $\arg z = \pi$, then l can be a curve belonging to the domain $\arg z \neq -\pi/2$ (i.e. this curve will not intersect the negative half of the imaginary axis). We denote the value of the argument of an arbitrary point in this domain (i.e. lying between $-\pi/2$ and $3\pi/2$) by θ; then $\theta = 0$ for the lower end of l and $\theta = \pi$ for the upper end. Since in the simply

connected domain $\arg z \neq -\pi/2$ the function $\varphi(z) = \ln|z| + i\theta$ is a primitive of $1/z$, according to Sec. 5.7

$$\int_l \frac{dz}{z} = \varphi(z) - \varphi(1) = \ln|z| + i\pi = \ln z.$$

Therefore, in all cases

$$F(z) = \int_1^z \frac{dz}{z} = \int_L \frac{dz}{z} = \ln z + 2n\pi i = \operatorname{Ln} z.$$

We see that the integral $\int_1^z (1/z)\,dz$ in the domain $z \neq 0$ represents the multiple-valued function $\operatorname{Ln} z$. The reader is advised to prove that the integral $\int_0^z \frac{dz}{1+z^2}$ in the domain $z \neq \pm i$ represents the multiple-valued function

$$\operatorname{Arc\,tan} z = \frac{1}{2i} \operatorname{Ln} \frac{i-z}{i+z}.$$

6

CAUCHY'S INTEGRAL FORMULA AND ITS IMPLICATIONS

6.1

CAUCHY'S INTEGRAL FORMULA

If $f(z)$ is a single-valued function analytic in a domain G and L is a rectifiable Jordan curve belonging to G together with its interior D (Fig. 44), *then for a point $z_0 \in D$ we have*

$$f(z_0)=\frac{1}{2\pi i}\int_L \frac{f(z)\,dz}{z-z_0}, \qquad (6.1)$$

where L is traversed in the positive direction, i.e. counterclockwise. This formula expresses the value of an analytic function at a point z of the interior of a closed curve in terms of the behavior of this function on the curve, and is called *Cauchy's integral formula.*

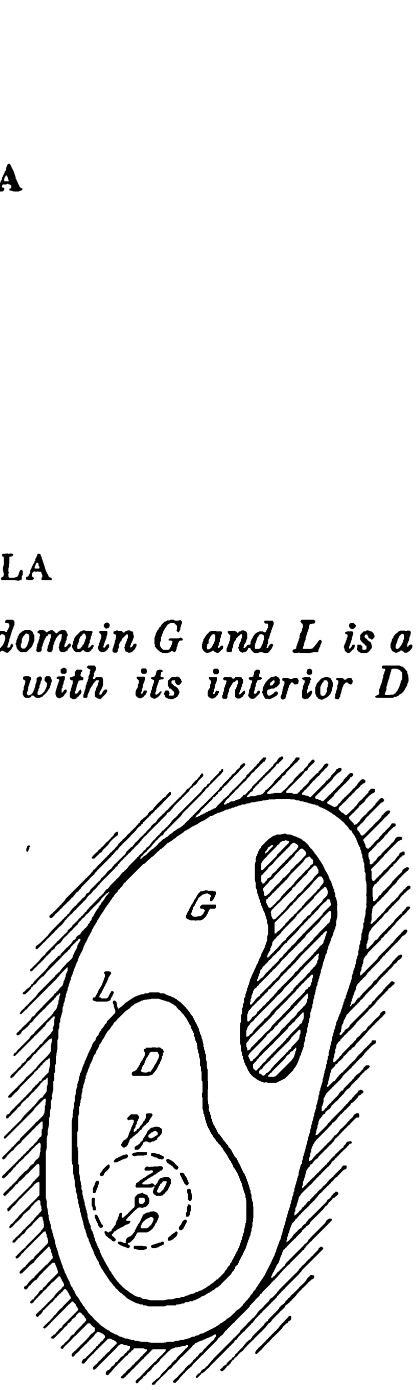

Fig. 44

To prove the above proposition, we take a circle γ_ρ centered at point z_0 and having a radius ρ so small that the circle $|z-z_0|\leqslant\rho$ lies inside L. Then for the composite contour formed by L and γ_ρ we have

$$\frac{1}{2\pi i}\int_L \frac{f(z)\,dz}{z-z_0}=\frac{1}{2\pi i}\int_{\gamma_\rho} \frac{f(z)\,dz}{z-z_0}. \qquad (6.2)$$

Hence, to verify (6.1) we need only prove that

$$f(z_0)=\frac{1}{2\pi i}\int_{\gamma_\rho} \frac{f(z)\,dz}{z-z_0}$$

or

$$\int_{\gamma_\rho} \frac{f(z)}{z-z_0}\,dz-2\pi i f(z_0)$$

$$=\int_{\gamma_\rho} \frac{f(z)}{z-z_0}\,dz-f(z_0)\int_{\gamma_\rho} \frac{dz}{z-z_0}=\int_{\gamma_\rho} \frac{f(z)-f(z_0)}{z-z_0}\,dz=0. \qquad (6.3)$$

In view of the continuity of $f(z)$ at z_0, the inequality

$$|f(z) - f(z_0)| < \varepsilon \quad (z \in \gamma_\rho)$$

is valid for any positive ε if $\rho < \delta(\varepsilon)$. Under these conditions we have

$$\left|\int_{\gamma_\rho} \frac{f(z)-f(z_0)}{z-z_0}\, dz\right| < \frac{\varepsilon}{\rho}\, 2\pi\rho = 2\pi\varepsilon.$$

Therefore

$$\lim_{\rho\to 0} \int_{\gamma_\rho} \frac{f(z)-f(z_0)}{z-z_0}\, dz = 0. \tag{6.4}$$

The integral to the right of the limit sign is exactly the left-hand side of Eq. (6.3) and therefore does not depend on ρ $\left(\text{the fact that } \int_{\gamma_\rho}\frac{f(z)}{z-z_0}\, dz \text{ does not depend on } \rho \text{ can be seen from (6.2)}\right)$. Hence, it vanishes for all values of ρ.

Thus, Eq. (6.3) is valid and so is Cauchy's integral formula (6.1).

The integral $J(z_0) = \frac{1}{2\pi i}\int_L \frac{f(z)}{z-z_0}\, dz$ is known as *Cauchy's integral*. We have calculated its value for a point z_0 that lies in the interior of the closed curve L. If z_0 is chosen in the exterior E of L, the integrand $f(z)/(z-z_0)$ is an analytic function not only on L but inside L as well (the denominator $z - z_0$ is nonzero on L and in the interior of L). According to Cauchy's integral theorem, Cauchy's integral is zero in this case. Hence,

$$J(z_0) = \frac{1}{2\pi i}\int_L \frac{f(z)\, dz}{z-z_0} = \begin{cases} f(z_0) & (z_0 \in D), \\ 0 & (z_0 \in E). \end{cases} \tag{6.5}$$

For $z_0 \in L$ Cauchy's integral loses its meaning not only as a proper integral but as an improper integral as well.

A particular case is when L is a circle with its center at z_0. If ρ is the radius of the circle, then the equation of the circle is $z = z_0 + \rho e^{i\varphi}$ $(0 \leqslant \varphi \leqslant 2\pi)$. Hence,

$$f(z_0) = \frac{1}{2\pi i}\int_L \frac{f(z)\, dz}{z-z_0} = \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(z_0+\rho e^{i\varphi})\, i\rho e^{i\varphi}}{\rho e^{i\varphi}}\, d\varphi$$

$$= \frac{1}{2\pi}\int_0^{2\pi} f(z_0+\rho e^{i\varphi})\, d\varphi. \tag{6.6}$$

The last integral can be considered the arithmetic mean of $f(z)$ on the circle $|z - z_0| = \rho$. Indeed, this integral is the limit, as

$n \to \infty$, of the arithmetic mean of the values of $f(z)$ at the vertices of a regular n-gon inscribed in the circle:

$$\frac{1}{2\pi}\int_0^{2\pi} f(z_0+\rho e^{i\varphi})\,d\varphi = \lim_{n\to\infty}\frac{1}{2\pi}\sum_{k=0}^{n-1} f\left(z_0+\rho e^{i\left(\varphi_0+\frac{2k\pi}{n}\right)}\right)\frac{2\pi}{n}$$

$$= \lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1} f\left(z_0+\rho e^{i\left(\varphi_0+\frac{2k\pi}{n}\right)}\right).$$

Thus, *the value of an analytic function $f(z)$ at any point z_0 in the analyticity domain is equal to the arithmetic mean of its values of any circle with center at z_0* (the only requirement is that both the circle and its interior lie in the same domain). From (6.6) it follows that real and imaginary parts of the analytic function, i.e. harmonic functions (see Sec. 2.13), also possess the property of the arithmetic mean:

$$u(x_0, y_0) = \frac{1}{2\pi}\int_0^{2\pi} u(x_0+\rho\cos\varphi, \quad y_0+\rho\sin\varphi)\,d\varphi,$$

$$v(x_0, y_0) = \frac{1}{2\pi}\int_0^{2\pi} v(x_0+\rho\cos\varphi, \quad y_0+\rho\sin\varphi)\,d\varphi.$$

6.2

EXPANDING AN ANALYTIC FUNCTION IN A POWER SERIES. LIOUVILLE'S THEOREM

In Sec. 4.3 we proved that the sum of a power series is an analytic function in the circle of convergence. Now with the help of Cauchy's integral formula we can prove that each analytic function in the circle of convergence can be expanded in a power series. More precisely, we will prove the following

Theorem. *Suppose that $f(z)$ is a single-valued function analytic in G. If $z_0 \in G$ and r is the distance from z_0 to the boundary of G, then in the circle $|z - z_0| < r$ the function $f(z)$ can be expanded in a series in powers of $z - z_0$.*

Let z be a point inside the circle. We will consider a concentric circle of radius ρ $(0 < \rho < r)$ containing point z in its interior (Fig. 45). If γ_ρ is the boundary of this second circle, then

$$f(z) = \frac{1}{2\pi i}\int_{\gamma_\rho}\frac{f(\zeta)\,d\zeta}{\zeta - z}. \tag{6.7}$$

To prove the theorem, it suffices to expand $f(\zeta)/(\zeta - z)$ into a power series in $z - z_0$ and then carry out termwise integration. We

have

$$\frac{1}{\zeta-z}=\frac{1}{\zeta-z_0-(z-z_0)}=\frac{1}{\zeta-z_0}\,\frac{1}{1-\dfrac{z-z_0}{\zeta-z_0}}=\sum_{0}^{\infty}\frac{(z-z_0)^n}{(\zeta-z_0)^{n+1}},$$

or

$$\frac{f(\zeta)}{\zeta-z}=\sum_{0}^{\infty}f(\zeta)\,\frac{(z-z_0)^n}{(\zeta-z_0)^{n+1}}.$$

For a fixed z this series uniformly converges with respect to

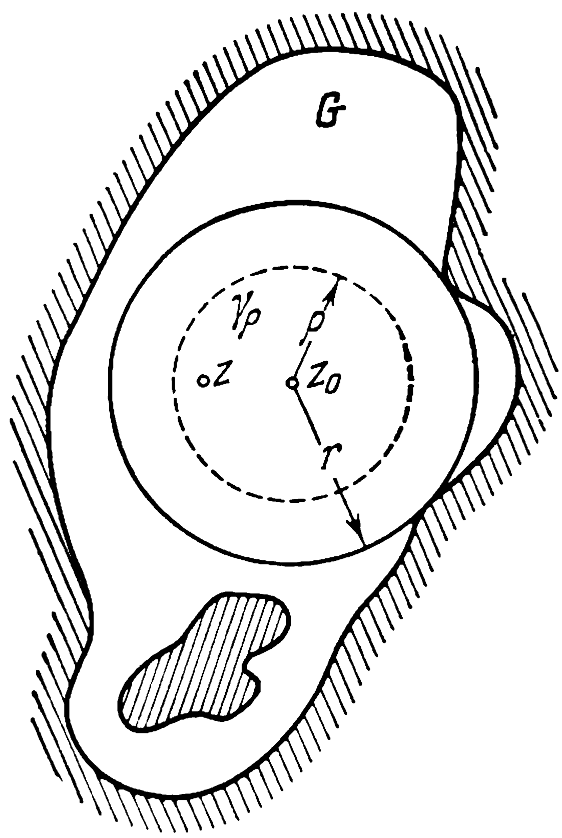

Fig. 45

$\zeta\in\gamma_\rho$, since

$$\left|f(\zeta)\,\frac{(z-z_0)^n}{(\zeta-z_0)^{n+1}}\right|\leqslant\max_{\gamma_\rho}|f(\zeta)|\,\frac{|z-z_0|^n}{\rho^{n+1}}$$

and the number series $\sum\limits_{0}^{\infty}\max\limits_{\gamma_\rho}|f(\zeta)|\,\dfrac{|z-z_0|^n}{\rho^{n+1}}$ converges (this is a geometric progression with the ratio $|z-z_0|/\rho<1$). Therefore, termwise integration is justified:

$$f(z)=\frac{1}{2\pi i}\int\limits_{\gamma_\rho}\sum_{0}^{\infty}f(\zeta)\,\frac{(z-z_0)^n}{(\zeta-z_0)^{n+1}}\,d\zeta$$

$$=\sum_{0}^{\infty}\frac{1}{2\pi i}\int\limits_{\gamma_\rho}\frac{f(\zeta)}{(\zeta-z_0)^{n+1}}\,d\zeta\,(z-z_0)^n.$$

The coefficients of the power series,

$$a_n = \frac{1}{2\pi i} \int\limits_{\gamma_\rho} \frac{f(\zeta)\,d\zeta}{(\zeta - z_0)^{n+1}} \quad (n = 0, 1, 2, \ldots),$$

do not depend on ρ. Indeed, if $\rho_1 \neq \rho$ and $0 < \rho_1 < r$, by the composite contour theorem we have

$$\int\limits_{\gamma_\rho} \frac{f(\zeta)\,d\zeta}{(\zeta - z_0)^{n+1}} = \int\limits_{\gamma_{\rho_1}} \frac{f(\zeta)\,d\zeta}{(\zeta - z_0)^{n+1}}.$$

Since the obtained expression

$$f(z) = \sum_0^\infty a_n (z - z_0)^n$$

has been established for an arbitrary point z in the circle $|z - z_0| < r$, we have arrived at the end of the proof.

Let us put $\max\limits_{\zeta \in \gamma_\rho} |f(\zeta)| = M(\rho)$; from the above formulas for the coefficients of the power series we have *Cauchy's inequalities*

$$|a_n| \leqslant \frac{1}{2\pi} \frac{M(\rho)}{\rho^{n+1}} 2\pi\rho = \frac{M(\rho)}{\rho^n} \quad (n = 0, 1, 2, \ldots), \tag{6.8}$$

which enable us to estimate the moduli of the coefficients of a power series from above in terms of the maximum of the modulus of the series sum of the circle $|z - z_0| = \rho$ and the radius of this circle.

For the remainder of the series we have the following estimate:

$$\left| \sum_{n+1}^\infty a_k (z - z_0)^k \right| \leqslant \sum_{n+1}^\infty \frac{M(\rho)}{\rho^k} |z - z_0|^k = \frac{M(\rho)\,|z - z_0|^{n+1}}{\rho^n (\rho - |z - z_0|)}. \tag{6.9}$$

It gives an idea of the error in using the approximate relation

$$f(z) \approx \sum_0^n a_k (z - z_0)^k.$$

By means of formula (6.8) we can now easily prove

Liouville's theorem. *Any entire function whose modulus is bounded is a constant.*

Proof. Suppose that $f(z)$ is an entire function. Then its expansion in the power series

$$f(z) = a_0 + a_1 z + a_2 z^2 + \ldots + a_n z^n + \ldots$$

converges in the entire plane (in this case the distance from point $z_0 = 0$ to the boundary of the finite plane, i.e. to point ∞, is infinite). If the modulus of $f(z)$ is bounded, i.e. $|f(z)| \leqslant M$, where M

is a positive constant, then

$$|a_n| \leqslant \frac{M(\rho)}{\rho^n} \leqslant \frac{M}{\rho^n} \quad (n=0, 1, 2, \ldots),$$

where ρ is any positive number. We fix $n \geqslant 1$ and tend ρ to ∞. This yields

$$|a_n| \leqslant 0, \text{ i.e. } a_n = 0 \ (n = 1, 2, 3, \ldots).$$

Hence

$$f(z) \equiv a_0,$$

which completes the proof.

As a simple example of applying Liouville's theorem we will prove

The fundamental theorem of algebra. *Every polynomial* $P(z) = c_0 + c_1 z + \ldots + c_n z^n$ $(n \geqslant 1, c_n \neq 0)$ *has at least one zero.*

Proof. We will prove the theorem by contradiction. Suppose that $P(z)$ has not a single zero. Then $f(z) = 1/P(z)$ is an entire function with $\lim\limits_{z \to \infty} f(z) = 0$. The modulus of $f(z)$ is bounded in the entire plane (indeed, there exists a positive R such that $|f(z)| < 1$ for $|z| > R$; if $\max\limits_{|z| \leqslant R} |f(z)| = m$, then $|f(z)| < m + 1$ for every z). Therefore, $f(z) \equiv \text{const} = 0$, which contradicts the definition of this function.

6.3

THE INFINITE DIFFERENTIABILITY OF ANALYTIC AND HARMONIC FUNCTIONS

From the results obtained in Sec. 6.2 we arrive at several important corollaries.

(i) *Every function $f(z)$ analytic in a domain G has derivatives of any order in G, i.e. is infinitely differentiable, in G.*

Indeed, $f(z)$ can be expanded in a neighborhood of any point in G in a power series, i.e. is infinitely differentiable (see Sec. 4.3).

Going by Sec. 4.3 we can write the expansion for $f(z)$ as

$$f(z) = \sum_{0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n \quad (|z - z_0| < r).$$

In the particular case where $f(z)$ is an entire function, the distance r from z_0 to the boundary of the finite plane (the point at infinity) is infinite. Assuming that $z_0 = 0$, we find that

$$f(z) = \sum_{0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n, \quad |z| < \infty.$$

If the reader calculates the derivatives of different orders for the elementary functions $\exp z$, $\sin z$, $\cos z$, $\sinh z$, and $\cosh z$, he will easily obtain the following expansions converging in the entire plane:

$$\exp z = \sum_0^\infty \frac{z^n}{n!}, \quad \sin z = \sum_1^\infty (-1)^{n-1} \frac{z^{2n-1}}{(2n-1)!},$$

$$\cos z = \sum_0^\infty (-1)^n \frac{z^{2n}}{(2n)!},$$

$$\sinh z = \sum_1^\infty \frac{z^{2n-1}}{(2n-1)!}, \quad \cosh z = \sum_0^\infty \frac{z^{2n}}{(2n)!}.$$

Let us dwell on the power expansions for branches of the simplest multiple-valued functions. We start with the branch $\ln z$ defined by the condition that $\ln 1 = 0$ in domain G whose boundary is the nonpositive part of the real axis: $x \leqslant 0$, $y = 0$ (this branch can be represented in the form $\ln z = \ln |z| + i \arg z$, where $|\arg z| < \pi$). Since $(\ln z)' = 1/z$, we find that $(\ln z)^{(k)} = (-1)^{k-1} \frac{(k-1)!}{z^k}$ and, hence, the expansion of $\ln z$ in powers of $z - 1$ is

$$\ln z = \sum_0^\infty \frac{(\ln z)^{(k)}_{z=1}}{k!} (z-1)^k,$$

or

$$\ln z = \sum_1^\infty (-1)^{k-1} \frac{(z-1)^k}{k}.$$

The distance r from point $z_0 = 1$ to the boundary of G is equal to unity. Hence, the above expansion can be used only when $|z - 1|$ is less than unity. Substituting z for $z - 1$, we arrive at a power series for $\ln(1 + z)$ that converges for $|z| < 1$:

$$\ln(1+z) = \sum_1^\infty (-1)^{k-1} \frac{z^k}{k} \quad (|z| < 1).$$

The function $\ln(1 + z)$ is the branch of $\mathrm{Ln}(1 + z)$ defined by the condition that $[\mathrm{Ln}(1 + z)]_{z=0} = \ln 1 = 0$ in domain G_1 whose boundary is the part of the real axis on which $x \leqslant -1$ $(y = 0)$.

We go on to the function z^α, where α is an arbitrary complex number. This is a multiple-valued function (when α is not an integer) that can be expressed in terms of the exponential and logarithmic functions: $z^\alpha = \exp(\alpha \,\mathrm{Ln}\, z)$.

We isolate the branch $\varphi(z)$ of this function in the same domain G as in the previous example by specifying that $\varphi(1) = 1$. We can express this branch in the form

$$\varphi(z) = \exp(\alpha \ln z),$$

whence

$$\varphi'(z) = \alpha \exp(\alpha \ln z) \frac{1}{z} = \alpha \exp(\alpha \ln z) \exp(-\ln z)$$
$$= \alpha \exp[(\alpha - 1) \ln z],$$

which implies that

$$\varphi^{(k)}(z) = \alpha(\alpha - 1) \ldots (\alpha - k + 1) \exp[(\alpha - k) \ln z] \quad (k = 1, 2, \ldots).$$

At point $z = 1$ the derivatives $\varphi^{(k)}(1) = \alpha(\alpha - 1) \ldots (\alpha - k + 1)$ and, hence, the expansion of $\varphi(z)$ in powers of $z - 1$ has the form

$$\varphi(z) = 1 + \sum_{1}^{\infty} \frac{\alpha(\alpha-1) \ldots (\alpha-k+1)}{k!} (z-1)^k.$$

This power series converges for $|z - 1| < 1$, just as in the previous example. Substituting z for $z - 1$, we arrive at the expansion of $\varphi(z + 1)$ in powers of z. If we use the more customary notation $(1 + z)^\alpha$ for $\varphi(z + 1)$ (which is understood as the function $\exp[\alpha \ln(1 + z)]$, which is single-valued in G_1), we arrive at the following expansion:

$$(1+z)^\alpha = 1 + \sum_{1}^{\infty} \frac{\alpha(\alpha-1) \ldots (\alpha-k+1)}{k!} z^k \quad (|z| < 1).$$

This is the well-known *binomial formula* for the most general case when the exponent is an arbitrary complex number.

(ii) *A derivative of an analytic function is an analytic function.* Indeed, $f'(z)$ has a derivative $[f'(z)]' = f''(z)$ everywhere in G.

(iii) *The real and imaginary parts of an analytic function have differentiable partial derivatives of any order.*

Indeed, if $f(z) = u(x, y) + iv(x, y)$ is analytic in G, then $u(x, y)$ and $v(x, y)$ are differentiable functions of x and y in G in view of the results obtained in Sec. 2.7. Since

$$f'(z) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i \frac{\partial u}{\partial y},$$

the partial derivatives of the first order of functions u and v are also differentiable in G. To prove corollary (iii) it suffices to show that for any nonnegative k and l the partial derivative $\frac{\partial^{k+l} u}{\partial x^k \, \partial y^l}$ $\left(\text{or } \frac{\partial^{k+l} v}{\partial x^k \, \partial y^l}\right)$

exists and is the real or imaginary part of a certain analytic function (namely, of $\pm f^{(k+l)}(z)$).

Let us suppose that this statement is true for all derivatives of order n, so that, for instance,

$$\frac{\partial^{k+l}u}{\partial x^k \partial y^l} = \operatorname{Re} F(z) = U(x, y), \quad \text{where } k+l=n.$$

Then there exist the following derivatives:

$$\frac{\partial U}{\partial x} = \frac{\partial^{k+l+1}u}{\partial x^{k+1} \partial y^l} = \operatorname{Re} F'(z)$$

and

$$\frac{\partial U}{\partial y} = \frac{\partial^{k+l+1}u}{\partial x^k \partial y^{l+1}} = -\operatorname{Im} F'(z),$$

from which follows the validity of the above statement in relation to $(n+1)$st-order partial derivatives and, hence, the validity of corollary (iii).

For one, there exist continuous (due to their differentiability) second-order partial derivatives of $u(x, y)$ and $v(x, y)$. It was this fact we relied on in Sec. 2.13 when proving that the real and imaginary parts of an analytic function are (conjugate) harmonic functions.

(iv) *If a function $f(z)$ is analytic in G and L is a rectifiable Jordan curve belonging to G together with its interior D, then at every point $z \in D$ and for every integer k the relationship*

$$f^{(k)}(z) = \frac{k!}{2\pi i} \int\limits_L \frac{f(\zeta)\, d\zeta}{(\zeta - z)^{k+1}} \tag{6.10}$$

is valid.

Indeed, if γ_r is a circle with its center at z and lies inside L, then

$$\frac{1}{2\pi i} \int\limits_L \frac{f(\zeta)\, d\zeta}{(\zeta - z)^{k+1}} = \frac{1}{2\pi i} \int\limits_{\gamma_r} \frac{f(\zeta)\, d\zeta}{(\zeta - z)^{k+1}} = a_k,$$

where a_k is the coefficient of $(z' - z)^k$ in the power series representing $f(z')$ in a neighbourhood of z. But this coefficient can be expressed in terms of the kth derivative of the series sum at point z thus:

$$a_k = \frac{f^{(k)}(z)}{k!}.$$

Comparing the last two formulas, we arrive at corollary (iv).

Note that Eq. (6.10) could be formally obtained from Cauchy's integral formula $f(z) = \frac{1}{2\pi i} \int\limits_L \frac{f(\zeta)\, d\zeta}{\zeta - z}$ if we were to differentiate both sides k times with respect to z; on the right-hand side the differentiation is done under the integral sign. In Sec. 6.6 we will give a proof of the validity of such differentiation on more general assumptions.

6.4
CHANGE OF VARIABLE UNDER THE INTEGRAL SIGN

To derive the rule for the change of variable in integrals of complex-valued functions we resort to a fact established in Sec. 6.3, namely, that the derivative $f'(z)$ of an analytic function is analytic and, hence, continuous.

Suppose that $f(z)$ is a function that is analytic in a domain G and L is a rectifiable curve in G. The function $w = f(z)$ maps L into a curve Γ that is also rectifiable. Indeed, if $z = \lambda(t)$, $\alpha \leqslant t \leqslant \beta$, is the equation for L, the equation for Γ is

$$w = f[\lambda(t)].$$

We divide $[\alpha, \beta]$ arbitrarily and denote the dividing points by $t_0 = \alpha$, $t_1, \ldots, t_n = \beta$ and put $z_j = \lambda(t_j)$ and $w_j = f[\lambda(t_j)]$. Then

$$\sum_{j=0}^{n-1} |w_{j+1} - w_j| = \sum_{j=0}^{n-1} \left| \int_{z_j}^{z_{j+1}} f'(z)\, dz \right|$$

$$\leqslant \max_L |f'(z)| \sum_{j=0}^{n-1} \int_{z_j}^{z_{j+1}} |dz| \leqslant \max_L |f'(z)| \times (\text{length of } L),$$

from which follows the rectifiability of Γ.

Let us show that

$$\int_\Gamma \Phi(w)\, dw = \int_L \Phi[f(z)]\, f'(z)\, dz,$$

for any function $\Phi(w)$ continuous on Γ. This formula expresses the rule of change of variable under the integral sign.

For proof we will consider the Riemann sums, whose limit is the integral $\int_\Gamma \Phi(w)\, dw$. We have

$$\sum_0^{n-1} \Phi(w_j)(w_{j+1} - w_j) = \sum_0^{n-1} \Phi[f(z_j)] \int_{z_j}^{z_{j+1}} f'(z)\, dz.$$

On the other hand,

$$\int_L \Phi[f(z)]\, f'(z)\, dz = \sum_0^{n-1} \int_{z_j}^{z_{j+1}} \Phi[f(z)]\, f'(z)\, dz,$$

which implies that

$$\sum_{0}^{n-1} \Phi(w_j)(w_{j+1}-w_j) - \int_L \Phi[f(z)]\, f'(z)\, dz$$

$$= \sum_{0}^{n-1} \int_{z_j}^{z_{j+1}} \{\Phi[f(z_j)] - \Phi[f(z)]\}\, f'(z)\, dz.$$

Let us assume that the subdivision of $[\alpha, \beta]$ is so fine that all the

$$\max_{t_j \leqslant t \leqslant t_{j+1}} |\Phi[f(z_j)] - \Phi[f(z)]|$$

can be made smaller than any positive ε. If we introduce the notation

$$\max_L |f'(z)| = M,$$

we have

$$\left| \sum_{0}^{n-1} \int_{z_j}^{z_{j+1}} \{\Phi[f(z_j)] - \Phi[f(z)]\}\, f'(z)\, dz \right| \leqslant M\varepsilon \times (\text{length of } L).$$

Therefore the Riemann sums

$$\sum_{0}^{n-1} \Phi[f(w_j)](w_{j+1}-w_j)$$

tend to

$$\int_L \Phi[f(z)]\, f'(z)\, dz,$$

as the subdivision becomes infinitely fine. This leads to the sought for relation.

6.5
MORERA'S THEOREM

Each function $f(z)$ that is analytic in a simply connected domain G is continuous in G and possesses the property that integrals of $f(z)$ along any path (in this domain) depend only on the lower and upper ends of specific path. Obviously this is simply a different formulation of Cauchy's integral theorem. This formulation, however, makes it possible to state the converse theorem.

Morera's theorem (1886). *If a function $f(z)$ is continuous in a simply connected domain G and the integral of this function along every curve depends only on the upper and lower ends of the curve, the function is analytic in G.*

We start the proof by noting that the integral $\int\limits_{z_0}^{z} f(z)\,dz$, where z_0 is fixed and z is the running point in G, is in this case a single-valued function $F(z)$. In Sec. 5.7 we proved, on assumptions that coincide with the present ones, that $F(z)$ is analytic and that $F'(z) = f(z)$, i.e. $f(z)$ is the derivative of an analytic function. But then by corollary (ii) of Sec. 6.3 it follows that $f(z)$ is analytic.

The hypothesis of Morera's theorem can be made less stringent. First, instead of requiring that the integral of $f(z)$ for any curve depend only on the upper and lower ends of this curve, it suffices to make sure that the integral along any triangular contour lying in G

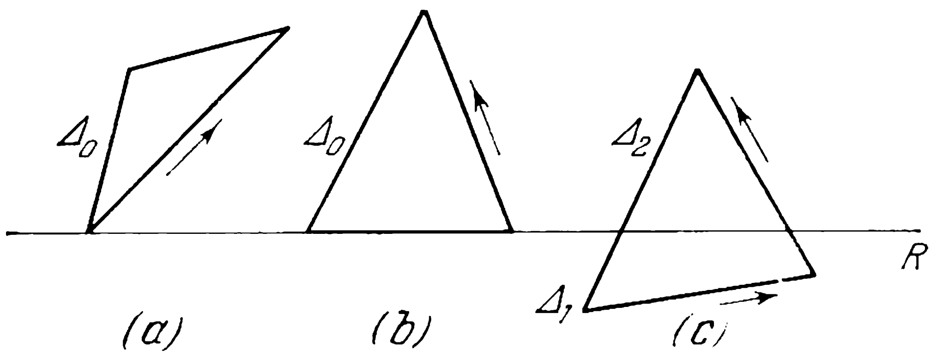

Fig. 46

is zero. Indeed, this proposition implies (and this follows from Cauchy's integral theorem) that the integral along any convex polygon and, hence, along any closed rectifiable curve is zero. But this fact is equivalent to $\int\limits_{L} f(z)\,dz$ being dependent only on the upper and lower ends for any L. Next, having in view a theorem we will prove in Sec. 9.5, we note that instead of requiring that the integral along any triangular contour be zero we can require that the integral be zero for triangular contours that have no common points with a given segment δ of a straight line R.

Indeed, if the triangular contour $\Delta_0 \subset G$ has common points with δ, then it either lies to one side of R (cases (a) and (b) in Fig. 46) or is dissected by R into two convex polygons Δ_1 and Δ_2 (case (c)). In each of these cases the function $f(z)$ is continuous on Δ_j ($j = 0$, 1, 2) and inside it and is analytic inside Δ_j (since inside Δ_j we can apply Morera's theorem to $f(z)$). As we have seen in Sec. 5.8, under such conditions

$$\int\limits_{\Delta_j} f(z)\,dz = 0.$$

Hence, for cases (a) and (b) we have

$$\int_{\Delta_0} f(z)\,dz = 0,$$

and for case (c)

$$\int_{\Delta_1} f(z)\,dz = 0 \quad \text{and} \quad \int_{\Delta_2} f(z)\,dz = 0,$$

which yields, after we add the two expressions,

$$\int_{\Delta_0} f(z)\,dz = 0.$$

Thus, we have proved that the integral along triangular contours intersected by R also vanishes. Therefore, $f(z)$ is analytic in the entire domain G (including the points of R).

6.6
WEIERSTRASS'S THEOREM ON UNIFORMLY CONVERGENT SERIES OF ANALYTIC FUNCTIONS

Suppose that the terms in the series $\sum_{1}^{\infty} f_n(z)$ *are single-valued functions analytic in a domain G and that this series converges uniformly in every closed circle belonging to G. Then* (a) *the sum* $f(z)$ *of the series is analytic in this domain,* (b) *the series can be differentiated termwise any number of times, i.e.* $\sum_{1}^{\infty} f_n^{(k)}(z) = f^{(k)}(z)$, *and* (c) *all the series obtained as a result of such differentiation converge uniformly in every closed circle belonging to G.*

In what follows, for the sake of brevity we will call a series that converges in every circle belonging to a domain G *uniformly convergent inside domain G.* Using the Heine-Borel theorem on coverings, we can easily prove that *uniform convergence inside a domain G is equivalent to uniform convergence on every closed, bounded point set belonging to G.* Note that uniform convergence inside G does not require that the series be convergent in the entire domain G. For this reason the theorem can be applied to any power series in its circle of convergence (see Sec. 4.4).

For the theorem to be valid it is essential that its hypothesis be satisfied for a domain *in* the plane. More precisely, it is essential that for each point of G there exists a circle with its center at that point and lying inside G. For instance, if we consider a series whose terms are the simplest analytic functions, polynomials $P_n(z)$ that

uniformly converge in a segment of the real axis (this segment constitutes a set of points for which the hypothesis is not satisfied), the sum of such a series may be not differentiable in the segment, even if it is, we may not be able to justify termwise differentiation of the series.* Therefore, Weierstrass's theorem expresses a specific property of analytic functions of a complex variable.

We start the proof by noting that it suffices to prove the theorem for an arbitrary circle $K : |z - z_0| < \rho$ that belongs to G together with its interior. To prove proposition (a) we use Morera's theorem. Since the series converges uniformly in $\overline{K}$ and its terms are continuous, its sum $f(z)$ is also continuous in K and the series can be termwise integrated along any triangular contour Δ lying in K. Whence

$$\int_{\Delta} f(z)\,dz = \sum_{1}^{\infty} \int_{\Delta} f_n(z)\,dz = 0$$

(we have employed the fact that the $f_n(z)$ is analytic in K). From Morera's theorem it then follows that $f(z)$ is analytic in K and, hence, it is analytic at all points of G.

Next, if $z \in K$, then

$$f^{(k)}(z) = \frac{k!}{2\pi i} \int_{|\zeta - z_0| = \rho} \frac{f(\zeta)}{(\zeta - z)^{k+1}}\,d\zeta$$

(see corollary (iv) in Sec. 6.3). But $\dfrac{f(\zeta)}{(\zeta - z)^{k+1}} = \sum\limits_{n=1}^{\infty} \dfrac{f_n(\zeta)}{(\zeta - z)^{k+1}}$, where the last series converges uniformly for a fixed z on the circle $|\zeta - z_0| = \rho$ (it is obtained from the uniformly convergent series $\sum\limits_{n=1}^{\infty} f_n(\zeta)$ by multiplying this by a function whose modulus is bounded, i.e. $\dfrac{1}{|\zeta - z|^{k+1}} \leqslant \dfrac{1}{(\rho - |z - z_0|)^{k+1}}$). We can, therefore perform termwise integration with respect to ζ and, hence,

$$f^{(k)}(z) = \frac{k!}{2\pi i} \int_{|\zeta - z| = \rho} \sum_{1}^{\infty} \frac{f_n(\zeta)}{(\zeta - z)^{k+1}}\,d\zeta$$

$$= \sum_{1}^{\infty} \frac{k!}{2\pi i} \int_{|\zeta - z_0| = \rho} \frac{f_n(\zeta)\,d\zeta}{(\zeta - z)^{n+1}} = \sum_{1}^{\infty} f_n^{(k)}(z),$$

i.e. we have shown that proposition (b) is valid, too.

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* See, for instance, G. M. Fikhtengol'ts, *A Course of Differential and Integral Calculus* [in Russian], vol. 2, Nauka, Moscow, 1966, p. 440.

Finally, to prove proposition (c), consider a circle $K' : |z - z_0| < \rho'$, where ρ' is greater than ρ but smaller than the distance from z_0 to the boundary of G. The closed circle $\overline{K}'$ belongs to G and contains the closed circle $\overline{K}$. For each point $z \in \overline{K}$ we have

$$\left| f^{(k)}(z) - \sum_1^N f_n^{(k)}(z) \right|$$

$$= \left| \frac{k!}{2\pi i} \int\limits_{|\zeta - z_0| = \rho'} \frac{f(\zeta)\, d\zeta}{(\zeta - z)^{k+1}} - \sum_1^N \frac{k!}{2\pi i} \int\limits_{|\zeta - z_0| = \rho'} \frac{f_n(\zeta)\, d\zeta}{(\zeta - z)^{k+1}} \right|$$

$$\leqslant \frac{k!}{2\pi} \int_0^{2\pi} \frac{\left| f(\zeta) - \sum_1^N f_n(\zeta) \right|}{|\zeta - z|^{k+1}} \rho'\, d\varphi$$

$$< \frac{k!}{2\pi} \frac{2\pi\rho'}{(\rho' - \rho)^{k+1}} \sup_{|\zeta - z_0| = \rho'} \left| f(\zeta) - \sum_1^N f_n(\zeta) \right|.$$

In view of the uniform convergence of $\sum_1^\infty f_n(\zeta)$ on the circle $|\zeta - z_0| = \rho'$ we can make the last factor arbitrarily small provided that N is sufficiently large (regardless of the position of $z \in \overline{K}$). Hence, for any $\varepsilon > 0$ and $z \in \overline{K}$ we have

$$\left| f^{(k)}(z) - \sum_1^N f_n^{(k)}(z) \right| < \varepsilon \text{ if } N > N_0(\varepsilon).$$

This completes the proof of Weierstrass's theorem.

In Sec. 6.2 we proved that a function $f(z)$ that is analytic in a circle $|z - z_0| < R$ can be expanded in this circle in a Taylor series. Since the power series converges uniformly in each concentric circle of a smaller radius, $|z - z_0| \leqslant r$ $(r < R)$, it follows that in such a circle the inequality

$$\left| f(z) - \sum_0^n \frac{f^{(k)}(z_0)}{k!} (z - z_0)^k \right| < \varepsilon, \quad n > N(\varepsilon, r),$$

holds for any $\varepsilon > 0$. In other words, *there exists a polynomial that approximates a function $f(z)$ that is analytic in circle $|z - z_0| \leqslant r$ with an arbitrarily high accuracy*. Now we will prove the converse proposition: *if a function $f(z)$ defined in a circle $|z - z_0| < R$ can be approximated arbitrarily accurately by a polynomial in each concentric circle of a smaller radius, the function is analytic in this circle.*

Indeed, suppose $\{r_n\}$ is a sequence of increasing positive numbers converging in R. For each circle $|z - z_0| \leqslant r_n$ we build a polyno-

mial $p_n(z)$ such that $|f(z) - p_n(z)| \leqslant 1/n$ for $|z - z_0| \leqslant r_n$. Then, as we can easily see, $f(z) = p_1(z) + \sum_{k=2}^{\infty} [p_k(z) - p_{k-1}(z)]$, i.e. $f(z)$ is represented in the circle $|z - z_0| < R$ by a polynomial series that converges uniformly in each smaller circle. Weierstrass's theorem then implies that $f(z)$ is analytic in $|z - z_0| < R$. Comparing these facts, we arrive at the following proposition:

A function $f(z)$ is analytic in the circle $|z - z_0| < R$ if and only if in each concentric circle of a smaller radius it can be approximated by a polynomial with an arbitrarily high accuracy.

We have thus established the validity of proposition (d) of Sec. 0.2.

6.7
THE COMPACTNESS PRINCIPLE

Suppose that $\{z_n\}$ is an arbitrary sequence of points. Then, as we know, in this sequence we can isolate a converging subsequence $\{z_{n_k}\}$. The limit of this subsequence can be an improper number, i.e. infinity. For the limit to be finite, it is sufficient that $\{z_n\}$ be bounded. Does this remain true for an arbitrary sequence of functions $\{f_n(z)\}$ analytic in a domain G? Can we be sure, for instance, that any such sequence of functions contains a subsequence of functions that converges uniformly in G (the limit functions must be also analytic in G, according to Weierstrass's theorem)? Simple examples show that for an arbitrary sequence of analytic functions there may not be any such subsequence.

Consider, for example, the sequence of functions $z, \ldots, nz, \ldots$ in the unit circle. It converges to zero at $z = 0$ and to ∞ at $z \neq 0$, and any subsequence contained in it has the same property. Now let us take the sequence $z, z^2, \ldots, z^n, \ldots$ in the circle $|z| < 2$. It converges uniformly to zero inside the unit circle and to infinity for $1 < |z| < 2$. Therefore, any subsequence contained in it has the same property. Obviously, in both cases there are no subsequences that uniformly converge in the corresponding domain (in the circle $|z| < 1$ in the first example and in the circle $|z| < 2$ in the second).

An infinite set E of functions that are analytic in a domain G is said to be *compact* in G if any sequence $\{f_n(z)\}$ of functions belonging to E contains a subsequence $\{f_n(z)\}$ that converges uniformly inside G. The above two examples show that the sequence $\{nz\}$ is noncompact in the unit circle and $\{z^n\}$ is noncompact in the circle $|z| < 2$.

Let us prove the following

Lemma. *If the sequence $\{f_n(z)\}$ is uniformly bounded in the circle $K: |z - z_0| < R$, i.e. there exists a positive number M such that the*

inequalities $|f_n(z)| < M$ *are valid at all points of* K *for any* n, *the sequence is compact in* K.

Proof. Consider the expansions of the $f_n(z)$ in power series:

$$f_n(z) = A_0^{(n)} + A_1^{(n)}(z - z_0) + \ldots + A_k^{(n)}(z - z_0)^k + \ldots \quad (6.11)$$

In view of Cauchy's inequalities for the coefficients of a power series (Sec. 6.2), we have for each ρ $(0 < \rho < R)$:

$$|A_k^{(n)}| \leqslant \frac{M_n(\rho)}{\rho^k}, \text{ where } M_n(\rho) = \max_{|z-z_0|=\rho} |f_n(z)| < M.$$

If ρ tends to the limit R, then for any nonnegative integer k we have

$$|A_k^{(n)}| \leqslant \frac{M}{R^k}, \quad n = 1, 2, \ldots . \quad (6.12)$$

This implies that any sequence of Taylor coefficients with a fixed index (i.e. with a fixed exponent of z) of the functions $f_n(z)$ is bounded. With this in mind we can select from the sequence $\{f_n(z)\}$ a subsequence

$$f_{n_1'}(z), f_{n_2'}(z), \ldots, f_{n_m'}(z), \ldots \quad (6.13)$$

such that the sequence $\{A_0^{(n_m')}\}$ of the absolute terms of the Taylor expansions for the functions in (6.13) converges to a certain limit. Then we select a new subsequence from the functions in (6.13),

$$f_{n_1''}(z), f_{n_2''}(z), \ldots, f_{n_m''}(z), \ldots, \quad (6.14)$$

such that in addition the sequence of the coefficients of the first-order terms in the Taylor expansions for the functions in (6.14) converges In general, if we select a subsequence

$$f_{n_1^{(m)}}(z), f_{n_2^{(m)}}(z), \ldots, f_{n_m^{(m)}}(z), \ldots \quad (6.15)$$

in a way such that the sequences of the coefficients of the zeroth-, first-, second-, ..., $(m - 1)$st-order terms in the corresponding Taylor expansions for the functions in (6.15) converge, the next step is to select a sequence of functions from (6.15),

$$f_{n_1^{(m+1)}}(z), f_{n_2^{(m+1)}}(z), \ldots, f_{n_m^{(m+1)}}(z), \ldots,$$

such that the sequence of coefficients of $(z - z_0)^m$ in the Taylor expansions converges to a finite limit.

Let us suppose that we have built subsequences (6.15) for any positive integer m. From these we select a *diagonal sequence* of functions that occupy the mth place in the mth sequence. We arrive at the sequence

$$f_{\nu_1}(z), f_{\nu_2}(z), \ldots, f_{\nu_m}(z), \ldots, \quad (6.16)$$

where $\nu_m = n_m^{(m)}$ and $f_{\nu_m}(z) = A_0^{(\nu_m)} + A_1^{(\nu_m)}(z - z_0) + \ldots + A_k^{(\nu_m)}(z - z_0)^k + \ldots$.

Subsequence (6.16) is obviously selected from functions in $\{f_n(z)\}$. Moreover, for any nonnegative integer k all functions in (6.16) starting from the $(k+1)$st function $f_{\nu_{k+1}}(z)$ belong to the sequence $f_{n_1^{(k+1)}}(z), f_{n_2^{(k+1)}}(z), \ldots, f_{n_k^{(k+1)}}(z), \ldots$. Hence, for any fixed nonnegative integer k the sequences $\{A_k^{(\nu_m)}\}$ of the coefficients in the Taylor expansions of the functions in (6.16) converge. Suppose that $\lim\limits_{m\to\infty} A_k^{(\nu_m)} = A_k$. Since each coefficient $A_k^{(\nu_m)}$ satisfies (6.12), the same holds for the A_k:

$$|A_k| \leqslant \frac{M}{R^k} \quad (k = 0, 1, 2, \ldots). \tag{6.17}$$

This implies that

$$\overline{\lim_{k\to\infty}} \sqrt[k]{|A_k|} \leqslant \frac{1}{R};$$

by the Cauchy-Hadamard theorem (Sec. 4.2) the radius of convergence of

$$A_0 + A_1(z - z_0) + \ldots + A_k(z - z_0)^k + \ldots \tag{6.18}$$

is not smaller than R. Whence the sum $f(z)$ of the series (6.18) is analytic in circle K. Let us show that the sequence (6.16) converges uniformly to $f(z)$ in K. For this it suffices to show that the sequence converges uniformly in each closed circle $|z - z_0| \leqslant \rho < R$.

Let us assume that ε is an arbitrary positive number. We select a positive integer n_0 such that

$$\sum_{n_0+1}^{\infty} M\left(\frac{\rho}{R}\right)^n < \frac{\varepsilon}{3}.$$

Then in view of (6.12) and (6.17) we have

$$\sum_{n_0+1}^{\infty} |A_n^{(\nu_m)}|\rho^n \leqslant \sum_{n_0+1}^{\infty} M\left(\frac{\rho}{R}\right)^n < \frac{\varepsilon}{3}$$

$$\text{and} \quad \sum_{n_0+1}^{\infty} |A_n|\rho^n \leqslant \sum_{n_0+1}^{\infty} M\left(\frac{\rho}{R}\right)^n < \frac{\varepsilon}{3},$$

and, hence, in $|z - z_0| \leqslant \rho$

$$|f_{\nu_m}(z) - f(z)| = \left|\sum_{0}^{\infty} (A_n^{(\nu_m)} - A_n)(z - z_0)^n\right| \leqslant \sum_{0}^{n_0} |A_n^{(\nu_m)} - A_n|\rho^n$$

$$+ \sum_{n_0+1}^{\infty} |A_n^{(\nu_m)}|\rho^n + \sum_{n_0+1}^{\infty} |A_n|\rho^n < \sum_{0}^{n_0} |A_n^{(\nu_m)} - A_n|\rho^n + \frac{2}{3}\varepsilon.$$

Since n_0 is fixed and $\lim\limits_{m\to\infty} A_n^{(\nu_m)} = A_n$, for sufficiently great $m > m_0$ we have

$$\sum_0^{n_0} | A_n^{(\nu_m)} - A_n | \rho^n < \frac{\varepsilon}{3}.$$

Consequently, $| f_{\nu_m}(z) - f(z) | < \varepsilon$ $(| z - z_0 | \leqslant \rho,\ m > m_0)$.

Hence, we have established the uniform convergence of the subsequence (6.16) in the circle K and found that sequence (6.11) contains such a subsequence. This completes the proof of the lemma.

To formulate the necessary and sufficient conditions for compactness in any domain G, we will call a set of functions E *uniformly bounded inside* G if for every closed point set F of G there exists a positive $M(F)$ such that each function $f(z) \in E$ obeys the inequality $| f(z) | \leqslant M(F)$ at all points of F. We will now prove

Montel's theorem (1907). *A set E of functions analytic in a domain G is compact if and only if it is uniformly bounded inside this domain.*

We will start by proving the necessity of the hypothesis for E to be compact. Indeed, if E is compact but not uniformly bounded inside G, there exists a closed set F of points in G on which the moduli of the functions belonging to E attain arbitrarily large values. In other words, for each positive integer n there exists a function $f_n(z) \in E$ and a point $z_n \in F$ such that at this point $| f_n(z_n) | > n$ $(n = 1, 2, 3, \ldots)$.

By virtue of the compactness of E, we can select from $\{f_n(z)\}$ a subsequence $\{f_{n_k}(z)\}$ that converges uniformly inside G and, in particular, on F. The limit function $f(z)$ is analytic in G and, hence, continuous on F. We denote $\max\limits_F | f(z) |$ by $M (< \infty)$. Since $| f_{n_k}(z) - f(z) | < 1$ for $n_k > N$ by virtue of uniform convergence at all points of F, we find that $| f_{n_k}(z) | \leqslant | f(z) | + 1 \leqslant M + 1$ at all points of F and for all $n_k > N$. But this contradicts the assumption that the $| f_{n_k}(z_{n_k}) |$ are arbitrarily large (at least exceed n_k). This shows that the hypothesis is necessary for E to be compact.

Now we will prove that it is sufficient. Suppose that E is uniformly convergent inside G. We must first see whether for every closed point set F in G we can select a uniformly convergent (on F) subsequence from a sequence $\{f_n(z)\}$ of functions belonging to E. To this end for each point $z \in F$ we take a neighborhood U_z that belongs (together with its boundary) to G and a neighborhood u_z of the same point whose radius is smaller than that of U_z (say by a factor of two). By the Heine-Borel theorem there exists a finite number of such neighborhoods, $u_{z_1}, u_{z_2}, \ldots, u_{z_q}$, that cover F. In each of the corresponding neighborhoods U_{z_j} $(j = 1, \ldots, q)$ the moduli of the func-

tions in $\{f_n(z)\}$ are uniformly bounded:

$$|f_n(z)| < M_j, \quad z \in U_{z_j} \quad (n = 1, 2, \ldots).$$

Then, according to the above lemma, out of $\{f_n(z)\}$ we can select a subsequence $\{f_{\nu'_k}(z)\}$ that converges uniformly inside U_{z_1}, out of $\{f_{\nu'_k}(z)\}$ a subsequence $\{f_{\nu''_k}(z)\}$ that converges uniformly inside U_{z_2}, etc., finally, out of $\{f_{\nu_k^{(q-1)}}(z)\}$ we can select a subsequence $\{f_{\nu_k^{(q)}}(z)\}$ that converges uniformly inside U_{z_q}. By virtue of construction, $\{f_{\nu_k^{(q)}}(z)\}$ converges uniformly inside each neighborhood U_{z_j} $(j = 1, \ldots, q)$ and, in fact, inside each neighborhood u_{z_j} $(j = 1, \ldots, q)$. Therefore, it converges uniformly on set F covered by the u_{z_j} $(j = 1, 2, \ldots, q)$.

We have thus proved that the sequence $\{f_n(z)\}$ for any closed set F contains a subsequence that converges uniformly on F. Let us consider a sequence of sets $\{F_\nu\}$ $(\nu = 1, 2, \ldots)$ each term F_ν of which is the set of points of G belonging to a closed circle $|z| \leqslant \nu$ with the distances of the points to the boundary of G being no less than $1/\nu$; each term, therefore, is a closed set (prove this). Obviously, for sufficiently large ν's $(\nu \geqslant \nu_0)$ all the F_ν are not empty; moreover, $F_{\nu+1} \supset F_\nu$ and, finally, each closed, bounded set $F \subset G$ belongs to all F_ν starting from a certain ν (if F lies in the circle $|z| < R$ and the distance from F to the boundary of G is $\rho > 0$, for $F \subset F_\nu$ it suffices to put $1/\nu < \rho_0 = \min(\rho, R^{-1})$). In other words, $\{F_\nu\}$ is an increasing sequence of closed sets that approaches G from within. Now, from $\{f_n(z)\}$ we select a subsequence $\{f_{n'_k}(z)\}$ that converges uniformly on F_1, from $\{f_{n'_k}(z)\}$ a subsequence $\{f_{n''_k}(z)\}$ that converges uniformly on F_2, etc. We arrive at the subsequences $\{f_{n_k^{(m)}}(z)\}$ $(m = 1, 2, \ldots)$, each of which (corresponding to the number m) is contained in the previous subsequence and converges uniformly on F_m. Obviously, the diagonal sequence

$$f_{n'_1}(z), f_{n''_2}(z), \ldots, f_{n_m^{(m)}}(z), \ldots$$

is contained in $\{f_n(z)\}$, and for any positive integer n all its terms starting from $f_{n_m^{(m)}}(z)$ belong to the subsequence $\{f_{n_k^{(m)}}(z)\}$. This implies that the diagonal sequence converges uniformly on any set F_m and, hence, is uniformly convergent inside G (since, as we noted above, each bound, closed set $F \subset G$ is contained in F_m with m sufficiently large). Thus, any sequence $\{f_n(z)\}$ on E contains a subsequence that converges uniformly inside G. This completes the proof of Montel's theorem.

Some branches of the theory of functions require notions that are more general than the notion of compactness considered here. We

will call a sequence $\{f_n(z)\}$ of functions that are analytic in G *uniformly converging to infinity inside domain* G if for every closed set $F \subset G$ and for any positive M there exists an $N(M)$ such that at all points of set F and for each $n > N(M)$ the inequalities

$$|f_n(z)| > M$$

are valid.

The set E of functions analytic in G is called a *normal family of functions in* G if every sequence $\{f_n(z)\}$ of functions belonging to E contains a subsequence $\{f_{n_k}(z)\}$ that converges uniformly inside G to an analytic function or to infinity.

In view of this definition each compact set of analytic functions is also a normal family; whence the condition that set E be uniformly bounded inside G, which is sufficient for E to be compact, is sufficient for E to be a normal family of functions in G. But the converse is not, in general, true (i.e. the condition is not necessary for E to be a normal family), which can easily be seen from the sequence $\{z+n\}$, which converges uniformly to ∞ in the unit circle and is therefore a normal family of functions.

6.8
THE UNIQUENESS THEOREM. VITALI'S THEOREM

An important property of analytic functions is expressed by

The uniqueness theorem. *There can be at most one single-valued function* $f(z)$ *analytic in domain* G *that admits given values on a point set* E *in* G *and that has at least one limit point* $z_0 \in G$.

Before proving the uniqueness theorem, we will study three examples.

(1) Suppose that G is the finite plane and E is the segment $0 < x < \pi/2$ on the real axis. Obviously, each point of E is a limit point for E and belongs to G. According to the theorem there exists at most one function $f(z)$ that is analytic in the finite plane and whose values for $z = x \in E$ coincide, for example with those of $\sin x$, i.e. $f(x) = \sin x$ $(0 < x < \pi/2)$.

We know that such a function does exist: $f(z) = \frac{e^{iz} - e^{-iz}}{2i} = \sin z$. The uniqueness theorem states that this is the only function that at points of E coincides with $\sin x$. A similar statement can be made for the function $\varphi(z) = \frac{e^{iz} + e^{-iz}}{2} = \cos z$.

(2) The uniqueness theorem shows that some relationships between analytic functions established under certain restrictions for the values of the independent variable are valid, in fact, without these restrictions. For instance, the validity of $\sin^2 z + \cos^2 z = 1$ for $0 < z = x < \pi/2$ follows directly from the Pythagorean theorem.

To show that it remains valid for any complex z, we need only to note that the function $F(z) = \cos^2 z + \sin^2 z$, which is analytic in the entire plane, admits values on E equal to unity. But the same values on E are admitted by $\Phi(z) \equiv 1$. By virtue of the uniqueness theorem $F(z)$ and $\Phi(z)$ must be the same analytic function, i.e. $\sin^2 z + \cos^2 z \equiv 1$ in the entire plane.

(3) Let us see whether there exists a function that is analytic in the entire finite plane G and for which $f\left(\frac{1}{n}\right) = f\left(-\frac{1}{n}\right) = \frac{1}{n}$ $(n = 1, 2, 3, \ldots)$. We denote the set of points $1, 1/2, 1/3, \ldots, 1/n, \ldots$ by E; obviously, E has a limit point 0 belonging to G. Since the analytic function $F(z) = z$ admits on E the given values $F(1/n) = 1/n$ $(n = 1, 2, 3, \ldots)$, there is no other function that satisfies the same conditions and, hence $f(z) \equiv F(z) \equiv z$. But at points $-1/n$ the function $f(z) = z$ is $-1/n$ instead of $1/n$. This shows that there is not a single function $f(z)$ that is analytic in G and yet for which

$$f\left(\frac{1}{n}\right) = f\left(-\frac{1}{n}\right) = \frac{1}{n} \quad (n = 1,\ 2,\ 3,\ \ldots).$$

(4) Finally, suppose that we wish to find a function $f(z)$ analytic in the finite plane for which $f(k\pi) = 0$ $(k = 0, \pm 1, \pm 2, \ldots)$. Here the values of the function are given on the set E of points $0, \pm\pi, \pm 2\pi, \pm 3\pi, \ldots$, which has not a single limit point in the finite plane. Consequently, we cannot rely on the uniqueness theorem. We can easily see that in this case the theorem is not valid. Indeed, there is an infinite number of analytic functions that satisfy the given conditions, for instance, $f(z) \equiv 0$, $\sin z$, $C \sin z$, $\sin^2 z$, $\sin^3 z, \ldots, \sin^n z$.

At present many classes of functions of a real and complex variable have been studied that are, generally speaking, nonanalytic but which possess the uniqueness property.

Examples. (a) Suppose that $P\ \{n_j\}$ is an increasing sequence of positive integers. In the interval $[a, b]$ of the real axis we consider the family of the functions $f(x)$ representable in this interval by the limit of the sequence $\{p_j(x)\}$ of polynomials such that $p_j(x)$ has a degree no greater than n_j and

$$|f(x) - p_j(x)| < Cq^{n_j}, \quad x \in [a, b],$$

where $C > 0$ and q, $0 < q < 1$, are constants that depend only on $f(x)$. S. N. Bernshtein proved that, for any two functions $f_1(x)$ and $f_2(x)$ of this family, the fact that they coincide in the interval $(\alpha, \beta) \subset [a, b]$ implies that they coincide in $[a, b]$. Such functions are said to be *quasi-analytic* (in the sense of Bernshtein). If the sequence $\{n_j\}$ grows faster than any geometric progression, the corresponding family may contain functions that are nowhere differentiable.

(b) Suppose that $\{\alpha_n\}$ is a sequence of points in the complex plane that may be dense everywhere in the plane or just in some region of the plane (e.g. all points whose two coordinates are rational numbers). It can be shown that series of the type $\sum_1^\infty \frac{A_n}{z-\alpha_n}$, where $\lim_{n\to\infty} \sqrt[n]{|A_n|} = 0$, converge everywhere in the plane except, possibly, a point set of measure zero (containing points from $\{\alpha_n\}$). A. Gonchar proved that the family of the functions representable in the form of the sums of series of the aforementioned type (provided that the sequence $\{\alpha_n\}$ is given and the expansion coefficients satisfy the above condition) possesses the property of uniqueness in the sense that the fact that any two functions of this family coincide on any arc of a continuous curve implies that they coincide everywhere in their domain of definition.

These families generalize the idea of monogenic functions studied by E. Borel (*monogenic* in the sense of Borel).

Let us now turn to the proof of the theorem. Suppose that $f(z)$ and $\varphi(z)$ are two analytic functions with the same values at all points in E. Then the difference $\psi(z) = f(z) - \varphi(z)$ is a function that is analytic in G and that vanishes at each point in E. If we can prove that $\psi(z) \equiv 0$, $z \in G$, then $f(z) \equiv \varphi(z)$, which is what the theorem states.

Consider first the particular case where G is a circle with a finite or infinite radius (in the latter case G coincides with the finite plane) centered at z_0, which is the limit point for E. The function $\psi(z)$, which is analytic in G and vanishes in E, can be represented by a power series that converges in the entire circle:

$$\psi(z) = c_0 + c_1(z - z_0) + \ldots + c_n(z - z_0)^n + \ldots \quad (6.19)$$

We have to show that all expansion coefficients are zeros, which implies that $\psi(z) \equiv 0$.

Since z_0 is the limit point for E, from E we can select a sequence of points $z_1, z_2, \ldots, z_k, \ldots$ that differ from z_0, are different, and converge to z_0. Then we have

$$\psi(z_k) = c_0 + c_1(z_k - z_0) + \ldots + c_n(z_k - z_0)^n + \ldots = 0 \quad (6.20)$$

$(k = 1, 2, 3, \ldots)$; as $z_k \to z_0$,

$$\psi(z_0) \to c_0 = 0.$$

Let us assume we have proved that all coefficients $c_0, c_1, \ldots, c_{n-1}$ are zeros. Then

$$\psi(z_k) = c_n(z_k - z_0)^n + c_{n+1}(z_k - z_0)^{n+1} + \ldots = 0,$$

or

$$c_n + c_{n+1}(z_k - z_0) + \ldots = 0 \quad (k = 1, 2, \ldots). \quad (6.21)$$

If in (6.21) we pass to the limit as $z_k \to z_0$, we find that

$$c_n = 0.$$

Hence, all the expansion coefficients in (6.19) are zeros, which means the theorem is true when G is a circle with the center at the limit point for E.

Now we will deal with the general case. Suppose K is a circle in G with its center at point z_0, which is the limit point for E. We have just proved that a function $\psi(z)$ that vanishes at each point of E lying in K vanishes everywhere in K. Since G does not coincide with K, there are points in G that do not belong to K. We wish to show that $\psi(z)$ vanishes at any such point z'. Connect z_0 with z' by a continuous curve L lying in G, and let $\rho > 0$ be the distance between L and the boundary Γ of G. We divide L into arcs by points $z_0, z_1, z_2, \ldots, z_{n-1}, z_n = z'$ in a way such that the distance between two neighboring points is smaller than ρ and draw circles K_j of radii ρ centered at each point z_j (Fig. 47). Obviously, the interior of K_j belongs to G and contains the center of the next circle, z_{j+1} $(j = 0, 1, 2, \ldots, n-1)$. At all points in K_0 the function $\psi(z)$ vanishes. Let us assume we have proved that $\psi(z) = 0$ at all points in K_j $(j \leqslant n-1)$. We wish to show that $\psi(z) = 0$ everywhere in K_{j+1}.

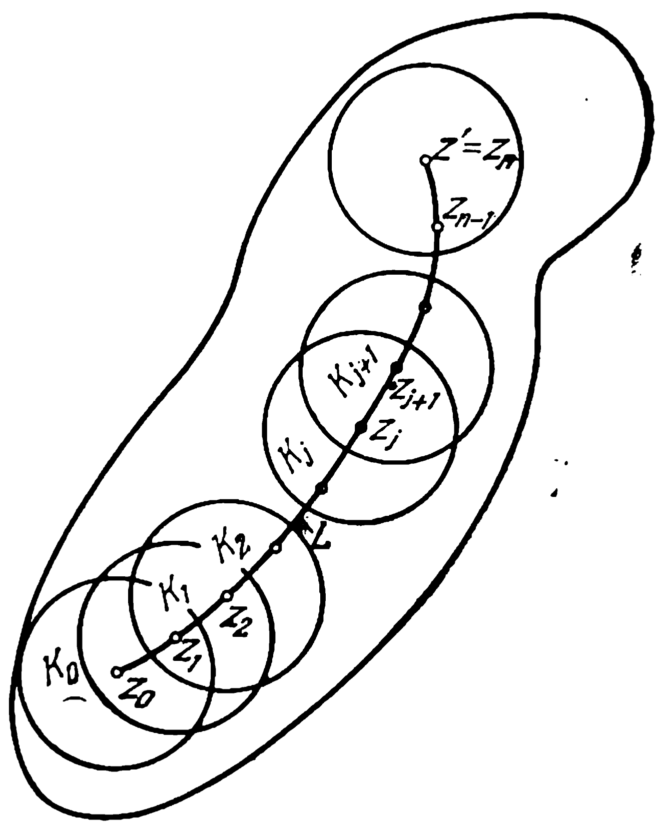

Fig. 47

Indeed, since the center z_{j+1} of circle K_{j+1} lies in K_j and, therefore, is a limit point for set on which $\psi(z) = 0$, the proved particular case of the theorem gives us $\psi(z) = 0$ in K_{j+1}. This implies that $\psi(z) = 0$ in K_n, too, and in fact at the center $z_n = z'$ of K_n. Thus, $\psi(z) \equiv 0$, $z \in G$, which completes the proof.

The uniqueness theorem establishes an important property of compact sets of analytic functions expressed by

Vitali's theorem (1903). *If a sequence $\{f_n(z)\}$ of functions analytic in a domain G is compact in G and converges on a point set $e \subset G$ that has at least one limit point belonging to G, then $\{f_n(z)\}$ converges uniformly inside G.*

Proof. In view of the compactness of $\{f_n(z)\}$, we can extract from any of its subsequences $\{f_{n'_k}(z)\}$ a subsequence $\{f_{n''_k}(z)\}$ that converges uniformly inside G. We wish to show that all subsequences of

$\{f_n(z)\}$ that converge uniformly inside G converge to the same limit function $f(z)$. Suppose $\lim_{k\to\infty} f_{\nu_k}(z) = f(z)$ and $\lim_{k\to\infty} f_{\mu_k}(z) = \varphi(z)$ (uniform convergence inside G). The functions $f(z)$ and $\varphi(z)$ are analytic in G (due to Weierstrass's theorem); moreover, $f(z) = \varphi(z)$ in e, where by hypothesis the given sequence $\{f_n(z)\}$ converges. Then the uniqueness theorem states that $f(z) = \varphi(z)$ everywhere in G.

Now we will prove that the entire sequence $\{f_n(z)\}$ is uniformly convergent inside G to $f(z)$. If this statement is untrue, we have a closed set $F \subset G$ on which $\{f_n(z)\}$ does not converge uniformly to $f(z)$. Then there must be a positive number α and indices n_k and corresponding points z_k of F such that the inequalities

$$|f_{n_k}(z_k) - f(z_k)| \geqslant \alpha\ (> 0), \quad k = 1, 2, \ldots, \tag{6.22}$$

are valid. Consider the sequence $\{f_{n_k}(z)\}$. It contains the subsequence $\{f_{n'_k}(z)\}$ that converges uniformly on F, and, as we have established earlier, the limit function is $f(z)$. Therefore

$$|f_{n'_k}(z) - f(z)| < \alpha \quad \text{for} \quad z \in F,\ n'_k > N$$

where in particular

$$|f_{n'_k}(z_k) - f(z_k)| < \alpha \tag{6.23}$$

for all sufficiently large k's. But inequalities (6.23) contradict (6.22). Hence, the assumption that $\{f_n(z)\}$ does not converge uniformly inside G to $f(z)$ is invalid. This completes the proof of Vitali's theorem.

6.9 RUNGE'S THEOREMS

Here we will prove that in any domain G a single-valued analytic function can be approximated by the limit of a sequence of rational functions that converges uniformly inside G. In the case where G is simply connected, these functions can be polynomials.

As a preliminary we will represent the domain G as the limit (or the union) of a strictly increasing sequence of domains G_n whose boundaries consist of finite numbers of polygonal contours. Let z_0 be a point belonging to G and let the square $|x - x_0| \leqslant \rho$, $|y - y_0| \leqslant \rho$ belong to G. We partition the plane into squares of infinitely diminishing sizes with the sides parallel to the coordinate axes; let point z_0 be a vertex of one of these squares. To build G_n we use squares whose sides are $\rho/2^n$ long and which belong to the square $|x - x_0| \leqslant n\rho$, $|y - y_0| \leqslant n\rho$.

From these we take only the squares that belong to G and the eight

immediately adjacent. Then the distance from each of these squares to the boundary of G is greater than $\rho/2^n$. The union of all such squares forms a bounded, closed subset of domains in G that breaks up into a finite number of connected components, i.e. domains that pairwise have no common points. For G_n we select the domain with z_0. Obviously, the boundary Γ_n of G_n consists of a finite number of polygons (in Fig. 48, G is a doubly connected unbounded domain and $n = 2$; the domain G_2 is hatched). When we substitute $n + 1$ for n, each square in the subdivision of the plane turns into four new squares. In view of the construction of G_n each point of the closed set $\overline{G}_n$ becomes an interior point of G_{n+1}, whence $\overline{G}_n \subset G_{n+1}$. Moreover, it is obvious that for each bounded, closed set $F \subset G$ we can find a positive $N(F)$ such that $F \subset G_n$ for $n > N(F)$.

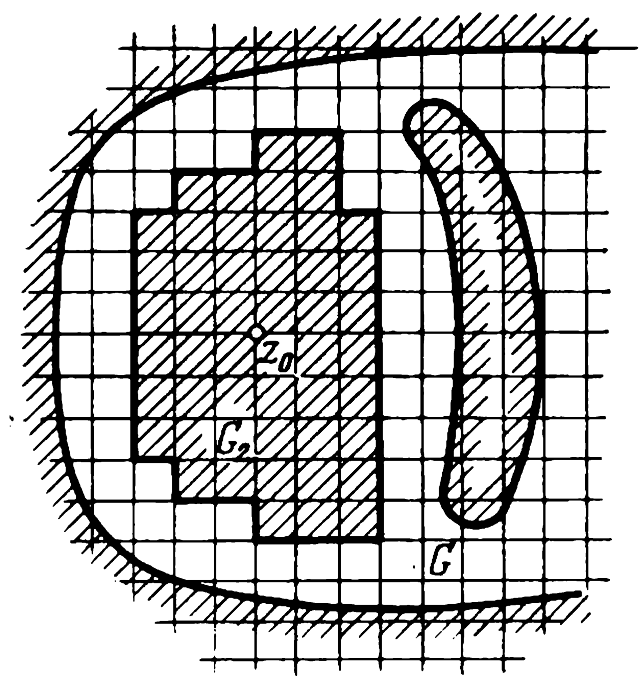

Fig. 48

Now let us formulate and prove **Runge's first theorem (1885).** *If $f(z)$ is single-valued and analytic in a domain G, then we can build a sequence of rational functions that uniformly converges to $f(z)$ inside G.*

Obviously, it suffices to prove the following, simpler, proposition: *for each closed domain $\overline{G}_m$ ($\overline{G}_m \subset G$) there exists a sequence of rational functions that uniformly converges to $f(z)$ in $\overline{G}_m$.* We know how to form a diagonal sequence (Sec. 6.7) and go over from the above case to a sequence that converges uniformly to $f(z)$ inside G.

Proof. Suppose G_m is one of the domains we have built above. Then $\overline{G}_m \subset G_{m+1}$ and, hence, at each point $z \in \overline{G}_m$,

$$f(z) = \frac{1}{2\pi i} \int\limits_{\Gamma_{m+1}} \frac{f(\zeta)}{\zeta - z}\, d\zeta,$$

where integration is carried out along the entire contour Γ_{m+1} bounding G_{m+1} (a composite contour, in general). Every Riemann sum $S(z)$ for this integral is the sum of a finite number of terms of the type

$$r_a(z) = \frac{1}{2\pi i} \frac{f(a)}{a - z} (a' - a),$$

where a and a' are boundary points of G_{m+1}; arcs σ with end points a and a' constitute, without gaps or repetitions, the entire boundary Γ_{m+1} of G_{m+1}.

Therefore, $f(z)$ can be viewed at each point $z \in \overline{G}_m$ as the limit of a sequence of rational functions $\{S(z)\}$. The uniform convergence of this sequence on $\overline{G}_m$ follows from it being uniformly bounded inside G_{m+1}, according to Vitali's theorem (Sec. 6.8). If δ is the distance between $\overline{G}_m$ and Γ_{m+1} and if we introduce the notation $M = \max\limits_{\Gamma_{m+1}} |f(\zeta)|$, then in $\overline{G}_m$ each function $r_a(z)$ satisfies the inequality

$$|r_a(z)| < \frac{1}{2\pi} \frac{M}{\delta} \sigma.$$

Whence $|S(z)|$ does not exceed $\frac{1}{2\pi} \frac{M}{\delta} \Sigma$ in $\overline{G}_m$, where Σ is the length of the composite contour Γ_{m+1}. This completes the proof of Runge's first theorem.

Let us dwell on the case of a simply connected domain G. Here the domains G_m, which we built earlier, are also simply connected. Indeed, if among the closed contours that constitute Γ_m we choose the outer contour Γ', its interior must all belong to G, i.e. must consist of the squares from which G_m is built. This means that Γ_m coincides with Γ', i.e. is a Jordan curve consisting of straight sections. In view of the above theorem, $f(z)$ can be represented in $\overline{G}_m$ in the form of the limit of a uniformly converging sequence of rational functions of the type $S(z) = \sum r_a(z)$ whose all poles (points a) lie on the contour Γ_{m+1} (also a Jordan curve); this last contour contains $\overline{G}_m$ strictly in its interior. We use the following lemma to substitute $S(z)$ by another rational function, $\widetilde{S}(z)$, that has only one pole ζ (of a sufficiently high order) in any preassigned point in the exterior of Γ_m.

Runge's lemma. *If D is a domain bounded by a Jordan curve and a and $\tilde{a}$ are two points lying in the exterior of L, then to every rational function $R(z) = P(z)/(z-a)^k$, where $P(z)$ is a polynomial of a degree not higher than k, and to every $\varepsilon > 0$ there corresponds a rational function of the type*

$$\widetilde{R}(z) = \frac{\widetilde{P}(z)}{(z-\tilde{a})^{\tilde{k}}},$$

where $\widetilde{P}(z)$ is a polynomial of a degree not higher than $\tilde{k}$, such that

$$|\widetilde{R}(z) - R(z)| < \varepsilon$$

everywhere in $\overline{D}$.

Proof. We connect points a and $\tilde{a}$ in the exterior of L by a broken line γ and denote the distance between L and γ by δ. We divide γ by

points $a_0 = a, a_1, \ldots, a_n = \tilde{a}$ into arcs $\sigma_1, \ldots, \sigma_n$ with lengths smaller than $\delta/2$. Now we build the first intermediate rational function with the only pole at a_1:

$$R_1(z) = R(z)\left[1 - \left(\frac{a-a_1}{z-a_1}\right)^n\right]^k = \frac{P_1(z)}{(z-a_1)^{k_1}}, \quad \text{where } k_1 = n_1 k.$$

The degree of $P_1(z)$ is no higher than $k+(n_1-1)k = k_1$. For $z \in \overline{D}$ we have

$$|R_1(z) - R(z)| = |R(z)| \left|\left[1 - \left(\frac{a-a_1}{z-a_1}\right)^{n_1}\right]^k - 1\right|$$
$$< M\left[\binom{k}{1}\left(\frac{\delta/2}{\delta}\right)^n + \binom{k}{2}\left(\frac{\delta/2}{\delta}\right)^{2n} + \ldots\right] < M\frac{2^k}{2^{n_1}},$$

where $M = \max\limits_{\overline{D}} |R(z)|$. If n_1 is sufficiently great, we have

$$|R_1(z) - R(z)| < \frac{\varepsilon}{n}, \quad z \in \overline{D}.$$

In a similar manner, we start from $R_1(z)$ and build the second intermediate function $R_2(z) = P_2(z)/(z-a_2)^{k_2}$ (where $k_2 = n_2 k_1$) such that

$$|R_2(z) - R_1(z)| < \frac{\varepsilon}{n}, \quad z \in \overline{D}.$$

Repeating this line of reasoning, we arrive on the nth step at the sought for function $\tilde{R}(z) = R_n(z)$.

Runge's second theorem. *Each function $f(z)$ that is single-valued and analytic in a simply connected domain G of the finite plane can be represented in the form of the limit of a sequence of polynomials uniformly converging inside G (or, which is the same, by a polynomial series uniformly converging inside G:*

$$f(z) = \sum_1^{\infty} P_n(z)).$$

Obviously, it suffices to prove that to each of the domains G_m, $\overline{G}_m \subset G$ and for every positive ε there exists a polynomial $Q(z)$ such that

$$|f(z) - Q(z)| < \varepsilon$$

everywhere in $\overline{G}_m$.

Proof. By Runge's first theorem there is a rational function $S(z) = \Sigma r_a(z)$ (with all its poles on the contour Γ_{m+1}) such that

$$|f(z) - S(z)| < \frac{\varepsilon}{3}.$$

Since points a lie in the exterior of G_m, we can apply Runge's lemma to each function $r_a(z)$. For the point $\tilde{a}$, to which the pole a is

shifted, we choose one point (independent of a) lying in the exterior of the circle $|z| < r$ containing $\overline{G}_m$ in its interior.

Then for each of the N terms $r_a(z)$ that constitute $S(z)$ we find a function $\tilde{r}_a(z)$ (with the only pole $\tilde{a}$) such that

$$|\tilde{r}_a(z) - r_a(z)| < \frac{\varepsilon}{3N}, \quad z \in \overline{G}_m.$$

Whence

$$\left| S(z) - \sum \tilde{r}_a(z) \right| < \frac{\varepsilon}{3}, \quad z \in \overline{G}_m,$$

or, if we put $\sum \tilde{r}_a(z) = R(z)$,

$$|f(z) - R(z)| < \frac{2\varepsilon}{3}, \quad z \in \overline{G}_m.$$

Note that $R(z)$ is a rational function with the only pole $z = \tilde{a}$, $|\tilde{a}| > r$. It can therefore be expanded in a power series, $R(z) = \sum_0^\infty c_n z^n$, that converges uniformly in the closed circle $|z| \leqslant r$ and, hence, on the closed set $\overline{G}_m$ belonging to this circle. The last step is to choose the partial sum of the above series: $\sum_0^\nu c_n z^n = Q(z)$, where ν is so large that

$$|R(z) - Q(z)| < \frac{\varepsilon}{3}, \quad |z| \leqslant r,$$

as the sought for polynomial. It obviously meets the required criteria. This completes the proof of the theorem.

Let us compare Weierstrass's theorem on uniformly convergent series and Runge's second theorem. In both cases we are dealing with uniformly convergent polynomial series. But when we are dealing with uniform convergence *within a one-dimensional* domain, i.e. in each segment of an interval on the real axis, the sum of the series is a continuous function, and we can obtain any continuous function by proper choice of the polynomials (Weierstrass's theorem). In the case of Runge's second theorem we are dealing with series of polynomials of one (complex) variable, and the series converge uniformly *inside a two-dimensional* simply connected domain. Here the sum may be only a function analytic in this domain, and as Runge's second theorem states we can obtain any analytic function by proper choice of the polynomials.

6.10

INTEGRALS DEPENDENT ON A PARAMETER

Weierstrass's theorem can be used to establish criteria for analyticity and differentiability under the integral sign of integrals that depend on a parameter.

Suppose that L is a rectifiable curve, G a domain in the complex plane, and $F(\zeta, z)$ a function that is defined for $\zeta \in L$ and $z \in G$ and that also satisfies the following conditions:

(1) $F(\zeta, z)$ is uniformly convergent if $\zeta \in L$ and z belongs to any closed circle $\overline{K}$ that belongs to G;

(2) for every $\zeta \in L$ the function $F(\zeta, z)$ is analytic with respect to z in G.

Under these conditions the integral

$$I(z) = \int_L F(\zeta, z)\, d\zeta$$

is an analytic function in G, and its derivative of order $p \geqslant 1$ is

$$I^{(p)}(z) = \int_L F_z^{(p)}(\zeta, z)\, d\zeta.$$

To prove this theorem we build a sequence $\{S_n(z)\}$ of Riemann sums that converges to $I(z)$:

$$S_n(z) = \sum_0^{n-1} F(\zeta_k^{(n)}, z)(\zeta_{k+1}^{(n)} - \zeta_k^{(n)}).$$

Under condition (2), the $S_n(z)$ are analytic functions in G. We wish to prove that the sequence converges uniformly inside G.

Under condition (1) for any $\varepsilon > 0$ and closed circle $\overline{K} \subset G$ there exists an $N(\varepsilon)$ such that at $n > N(\varepsilon)$ we have

$$|F(\zeta, z) - F(\zeta_k^{(n)}, z)| < \varepsilon, \quad k = 0, 1, \ldots, n-1,$$

where z is any point belonging to $\overline{K}$, and ζ is any point on an arc obtained by subdivision of L and having end points $\zeta_k^{(n)}$ and $\zeta_{k+1}^{(n)}$. Then (i.e. for $n > N(\varepsilon)$ and any $z \in \overline{K}$)

$$|I(z) - S_n(z)| = \left| \int_L F(\zeta, z)\, d\zeta - \sum_{k=0}^{n-1} \int_{\zeta_k^{(n)}}^{\zeta_{k+1}^{(n)}} F(\zeta_k^{(n)}, z)\, d\zeta \right|$$

$$= \left| \sum_{k=0}^{n-1} \int_{\zeta_k^{(n)}}^{\zeta_{k+1}^{(n)}} [F(\zeta, z) - F(\zeta_k^{(n)}, z)]\, d\zeta \right| \leqslant \varepsilon \text{ (length of } L),$$

whence follows the uniform convergence of $\{S_n(z)\}$.

From Weierstrass's theorem we conclude that, first, $I(z)$ is analytic in G and, second, for any positive integer p the sequence

$$\{S_n^{(p)}(z)\} = \left\{ \sum_{k=0}^{n-1} \frac{\partial^p F(\zeta_k^{(n)}, z)}{\partial z^p} (\zeta_{k+1}^{(n)} - \zeta_k^{(n)}) \right\}$$

converges uniformly to $I^{(p)}(z)$ inside G. This implies that $\int\limits_L \frac{\partial^p F(\zeta, z)}{\partial z^p} d\zeta$ exists and is equal to $I^{(p)}(z)$.

Consider, for example, the *integral of the Cauchy type*

$$I(z) = \frac{1}{2\pi i} \int\limits_L \frac{\varphi(\zeta)\, d\zeta}{\zeta - z},$$

where $\varphi(\zeta)$ is a function continuous on L. Suppose that G is one of the domains into which L partitions the complex plane (if L is a segment of a straight line, G is its exterior; if L is the lemniscate of Bernoulli (the form of a fallen "eight"), G is any of the three domains into which L partitions the plane; etc.). In our case $F(\zeta, z) = \varphi(\zeta)/ /(\zeta - z)$ and, obviously, conditions (1) and (2) are met. For this reason we can state that $I(z)$ is a function analytic in G and, for any positive integer p

$$I^{(p)}(z) = \frac{p!}{2\pi i} \int\limits_L \frac{\varphi(\zeta)\, d\zeta}{(\zeta - z)^{p+1}}.$$

If L is a rectifiable Jordan curve, G the interior of L, and $\varphi(z)$ a function that is analytic in a domain containing G and the boundary of G, the integral of the Cauchy type is simply Cauchy's integral.

We have arrived once more at formulas for the derivatives of Cauchy's integral.

Here are some examples of integrals of the Cauchy type that are not Cauchy's integrals:

$$\text{(a)}\ \frac{1}{2\pi i} \int\limits_{-1}^{1} \frac{d\zeta}{\zeta - z}, \quad \text{(b)}\ \frac{1}{2\pi i} \int\limits_{|\zeta|=1} \frac{\zeta^{-1}\, d\zeta}{\zeta - z}.$$

The reader can easily verify that in the case (a) the integral represents the function $\frac{1}{2\pi i} \ln \frac{z-1}{z+1}$ in the exterior of the segment $[-1, 1]$, whereas in the case (b) the integral, which is obviously

$$\frac{1}{z}\left[\frac{1}{2\pi i} \int\limits_{|\zeta|=1} \frac{d\zeta}{\zeta - z} - \frac{1}{2\pi i} \int\limits_{|\zeta|=1} \frac{d\zeta}{\zeta}\right],$$

represents $-1/z$ in the exterior of the unit circle and 0 inside this circle.

Vitali's theorem (Sec. 6.8) makes it possible to prove that the main proposition in this section, condition (1) (uniform convergence in ζ and z), can be changed to another proposition valid under more general assumptions:

(1') *for every $z \in G$ the function $F(\zeta, z)$ is continuous in ζ on L, and for every closed circle $\overline{K}$ its modulus is uniformly bounded in $\zeta \in L$.*

Indeed, if condition (1′) is met, the sequence of the Riemann sums

$$S_n(z)=\sum_{0}^{n-1} F(\zeta_k^{(n)}, z)(\zeta_{k+1}^{(n)}-\zeta_k^{(n)}),$$

is, obviously, uniformly bounded in modulus inside G. Since this sequence converges to $I(z)$ in this domain, Vitali's theorem (in using this theorem we rely on condition (2), of course) states that the sequence converges uniformly inside G.

An important case from the standpoint of applications is where L is an unlimited curve: $\zeta=\lambda(t)$, $\alpha\leqslant t<\beta$, $\lim\limits_{t\to\beta}\lambda(t)=\infty$, where every arc L_τ, $\alpha\leqslant t\leqslant\tau<\beta$, of the curve is rectifiable. An example is the *Laplace transform of* $\varphi(t)$:

$$\int_0^\infty e^{-zt}\varphi(t)\,dt.$$

Here integration is carried out along the positive half of the real axis ($L:\zeta=t$, $0\leqslant t<\infty$) and $\varphi(t)$ is a function of a real variable defined and continuous for $0\leqslant t<\infty$.

Let us consider a similar integral in the general form

$$I(z)=\int_L F(\zeta, z)\,d\zeta.$$

We can easily see that the main theorem of this section remains valid if for condition (1) we substitute the following condition:

(1″) for every $z\in G$ the function $F(\zeta, z)$ is continuous in ζ on L; for every closed circle $\overline{K}\subset G$ and for every arc L_τ, $\alpha<\tau<\beta$, it is bounded in modulus:

$$|F(\zeta, z)|\leqslant M(\tau, \overline{K}),\quad \zeta\in L_\tau,\quad z\in\overline{K};$$

finally, the integral $I(z)$ is absolutely convergent for every $z\in G$:

$$\lim_{\tau\to\beta}\int_{L_\tau}|F(\zeta, z)|\,|d\zeta|=\int_L|F(\zeta, z)|\,|d\zeta|,$$

where the function $\int_L|F(\zeta, z)|\,|d\zeta|$ is bounded inside G.

In the case of the Laplace transform the only condition is that $\varphi(t)$ be continuous and satisfy the condition

$$C=\overline{\lim_{t\to\infty}}\frac{\ln|\varphi(t)|}{t}<+\infty$$

and that G be the half-plane $x>C$ (so that conditions (1″) and (2) are met). The reader is advised to verify the validity of this proposition.

6.11

A-POINTS. THE MAXIMUM MODULUS PRINCIPLE

Suppose that A is an arbitrary finite complex number. We call the roots of the equation $f(z) = A$, where $f(z)$ is analytic in a domain G, the *A-points of $f(z)$*. It follows from the uniqueness theorem that when $f(z) \not\equiv A$, the set of all the A-points has no single limit point (otherwise we would find that $f(z) \equiv A$). This implies that *every bound, closed set F in G can contain only a finite number of A-points (for A fixed)*.

Indeed, if we suppose that F contains an infinite number of A-points, we find that this set of A-points has a limit point belonging to $\overline{F}$ and, hence, to G.

Let us assume that z_0 is an A-point of $f(z)$, so that $f(z_0) = A$. In a neighborhood of z_0 we have

$$f(z) = A + f'(z_0)(z - z_0) + \frac{f''(z_0)}{2!}(z - z_0)^2 + \dots,$$

or

$$f(z) - A = f'(z_0)(z - z_0) + \frac{f''(z_0)}{2!}(z - z_0)^2 + \dots \tag{6.24}$$

If $f(z) \not\equiv A$, then some of the coefficients in the series on the right-hand side of (6.24) may be nonzero. Suppose the coefficient of $(z - z_0)^k$ is the first nonzero coefficient. Then instead of (6.24) we have

$$f(z) - A = (z - z_0)^k \left[\frac{f^{(k)}(z_0)}{k!} + \frac{f^{(k+1)}(z_0)}{(k+1)!}(z - z_0) + \dots\right], \tag{6.25}$$

where $f^{(k)}(z_0) \neq 0$. We call the positive integer k the *order of the A-point* z_0 and z_0 an *A-point of the kth order*; if $k = 1$, the A-point is said to be *simple*.

Due to its definition a simple A-point is characterized by the fact that for it $f(z_0) = A$ and $f'(z_0) \neq 0$; an A-point of an order greater than 1 is characterized by the fact that

$$f(z_0) = A, \quad f'(z_0) = 0, \quad \dots, \quad f^{(k-1)}(z_0) = 0, \quad f^{(k)}(z_0) \neq 0.$$

What we have just said is, of course, applicable to the case where $A = 0$; for the sake of brevity the zero points of an analytic function are called its *zeros*. We can easily see that the definition of the order of a zero of a polynomial or, in general, any rational function is consistent with the general definitions given above. Note that for each $A \neq 0$ the A-points of a function $f(z)$ are the zeros of $f(z) - A$.

Examples. (a) For $\sin z$ all its zeros $z = k\pi$ $(k = 0, \pm 1, \pm 2, \dots)$ are simple because $(\sin z)' = \cos z$ does not vanish at $z = k\pi$.

(b) For $\cos z$ all the 1-points $z = 2k\pi$ $(k = 0, \pm 1, \pm 2, \dots)$ are of order 2 because $(\cos z)' = -\sin z = 0$ at $z = 2k\pi$ whereas

$(\cos z)'' = -\cos z \neq 0$ at $z = 2k\pi$. This implies that all the zeros of $\cos z - 1$ (or $1 - \cos z$) are of order 2.

(c) For $f(z) = \sin z - z$ the point $z = 0$ is a zero of order 3 because $f(0) = 0$, $f'(0) = 0$, $f''(0) = 0$, but $f'''(0) = -\cos 0 \neq 0$.

If a function $f(z)$ that is analytic in a domain G and not a constant takes on at a point $z_0 \in G$ the value A ($f(z_0) = A$), then, as we have just seen, all values of $f(z)$ will differ from A in a small neighborhood U of z_0. More than that, there are always values of $f(z)$ whose moduli are greater than $|A|$ and, if $A \neq 0$, smaller than $|A|$. In other words, the modulus of an analytic function that is not a constant can have extrema only at the zero points of this function (where it attains its minima). In particular, the modulus of $f(z)$ cannot remain constant in any subdomain of G.

This fundamental property of analytic functions is expressed by

The maximum modulus principle. *When a function $f(z)$ is analytic and not a constant in a domain G of the complex plane, $|f(z)|$ never attains its maximum in G.*

In formulating the theorem we did not mention the minimum of the modulus, because if $f(z)$ has no zeros in G, the function $1/f(z)$ is also analytic and the minima of $|f(z)|$ would correspond to the maxima of $|1/f(z)|$. On the other hand, the following line of reasoning can be extended to the case of minima provided that $f(z)$ has no zeros.

We will prove the theorem by contradiction. Suppose that $|f(z)|$ has a maximum at point $z_0 \in G$. Obviously, $f(z_0) = A \neq 0$ (otherwise $f(z) = 0$ in some neighborhood of z_0 and, hence, $f(z)$ is identically zero, which contradicts the hypothesis of the theorem).

We expand $f(z)$ in a power series in $z - z_0$:

$$f(z) = A + a_k (z - z_0)^k + a_{k+1} (z - z_0)^{k+1} + \ldots,$$

where $a_k \neq 0$ and $k \geqslant 1$. On the strength of our assumption in a sufficiently small neighborhood U of point z_0 the inequality $|f(z)| \leqslant |A|$ is valid. We take point $z_1 \in U$ that differs from z_0 and lies on one of the rays

$$\operatorname{Arg}[a_k (z - z_0)^k] = \operatorname{Arg} A,$$

i.e. $\operatorname{Arg}(z - z_0) = \frac{1}{k} \operatorname{Arg} \frac{A}{a_k}$ (there is exactly k distinct rays). Then the vectors that represent A and $a_k (z_1 - z_0)^k$ are parallel and point in the same direction, whence

$$|A + a_k (z_1 - z_0)^k| = |A| + |a_k (z_1 - z_0)^k|.$$

In addition we require that z_1 be so close to z_0 on the selected ray that

$$|a_{k+1} (z_1 - z_0)^{k+1} + a_{k+2} (z_1 - z_0)^{k+2} + \ldots| < \frac{1}{2} |a_k (z_1 - z_0)^k|$$

(here we rely on the continuity property of the sum of the power series $a_{k+1}(z-z_0)+a_{k+2}(z-z_0)^2+\ldots$ at point $z=z_0$).

Then at all such points z_1 (arbitrarily close to z_0) we have

$$\begin{aligned}|f(z_1)| &= |A+a_k(z_1-z_0)^k+a_{k+1}(z_1-z_0)^{k+1}+\ldots| \\ &\geqslant |A+a_k(z_1-z_0)^k|-|a_{k+1}(z_1-z_0)^{k+1}+\ldots| \\ &= |A|+|a_k(z_1-z_0)^k|-|a_{k+1}(z_1-z_0)^{k+1}+\ldots| \\ &> |A|+\frac{1}{2}|a_k(z_1-z_0)^k|>|A|.\end{aligned}$$

But this contradicts the assumption. The proof of the maximum modulus principle is complete.

Suppose that $f(z)$ is not a constant and is continuous in a closed domain $\overline{G}$ and analytic in G. Then its modulus, which is a function continuous in $\overline{G}$, must attain its upper bound at some point $\zeta\in\overline{G}$. In view of the maximum modulus principle, this point cannot belong to G; hence, it is a point on the boundary of G. Thus, *if a function $f(z)$ is not a constant, is continuous in a closed domain $\overline{G}$, and is analytic in G, then $|f(z)|$ attains its maximum on the boundary of G.*

Obviously, this proposition remains valid when $f(z)$ is a constant. In Sec. 10.2 we will prove the maximum modulus principle in a way that differs from the above proof.

6.12

POWER SERIES EXPANSIONS FOR ANALYTIC FUNCTIONS

Let us consider some methods of expanding analytic functions into power series. In principle the expansion coefficients of a Taylor series are defined via the formulas

$$a_n=\frac{f^{(n)}(z_0)}{n!}\quad(n=0,\ 1,\ 2,\ \ldots).$$

But the actual computations based on calculating the successive derivatives of $f(z)$ can be very tedious. In many cases of practical importance, however, we can arrive at the Taylor series by starting from other well-known expansions.

Suppose $f(z)$ is represented by a series of analytic functions converging uniformly inside a neighborhood $|z-z_0|<\rho$ of point z_0:

$$f(z)=\sum_1^\infty f_n(z).$$

Then in view of Weierstrass's theorem we have

$$f^{(k)}(z)=\sum_1^\infty f_n^{(k)}(z)$$

and

$$\frac{1}{k!} f^{(k)}(z_0) = \sum_1^{\infty} \frac{f_n^{(k)}(z_0)}{k!}.$$

Here $\frac{1}{k!} f_n^{(k)}(z_0)$ is the coefficient of $(z - z_0)^k$ in the Taylor series for $f_n(z)$, and $\frac{1}{k!} f^{(k)}(z_0)$ is the coefficient of $(z - z_0)^k$ in the Taylor series for $f(z)$.

Hence, *the Taylor expansion coefficients for the sum of a uniformly convergent series of analytic functions,* $\sum_1^{\infty} f_n(z)$, *are obtained by adding the respective Taylor expansion coefficients (i.e. coefficients of* $(z - z_0)^k$ *in the kth term of each series) for each function* $f_n(z)$.

Examples. (a) Consider the series

$$F(z) = \sum_1^{\infty} \frac{z^n}{1-z^n}.$$

The terms are functions of z that are analytic in the unit circle, and $\sum_1^{\infty} \frac{z^n}{1-z^n}$ converges uniformly inside the unit circle. Indeed, if E is a closed point set in this circle and $\delta > 0$ is the distance from E to the circumference of the unit circle, then for any point $z \in E$ we have $|z| \leqslant 1 - \delta = \rho < 1$; hence, $\left|\frac{z^n}{1-z^n}\right| \leqslant \frac{\rho^n}{1-\rho^n} \leqslant \frac{\rho^n}{1-\rho}$. Since the series $\sum_1^{\infty} \frac{\rho^n}{1-\rho}$ converges (this is a geometric progression with the ratio ρ), the series under consideration converges uniformly on E, i.e. converges uniformly inside the unit circle. To determine the Taylor expansion coefficient of z^k for $F(z)$, we must add, according to what we have established, the corresponding Taylor expansion coefficients of z^k for each of the functions

$$\frac{z^n}{1-z^n} = z^n + z^{2n} + z^{3n} + \ldots \quad (n = 1, 2, \ldots).$$

The coefficient of z^k in this series is zero if k is not divisible by n and is unity if it is. Hence, the sought for coefficient of z^k in the Taylor expansion for $F(z)$ is simply a sum of 1's, with the number of such terms being equal to that of all the positive integral divisors of k. Denoting this number by $\tau(k)$ ($\tau(1) = 1$, $\tau(2) = 2$, $\tau(3) = 2$,

$\tau(4) = 3$, $\tau(5) = 2, \ldots$), we have

$$F(z)=\sum_{1}^{\infty} \tau(k)\, z^k.$$

This is the sought for expansion. Since by Weierstrass's theorem $F(z)$ is a function analytic in the unit circle, the above series converges in the unit circle.

Note that at $z = 1$ the series diverges because it assumes the form $\sum_{1}^{\infty} \tau(k)$, with all terms being positive integrals. This means that the radius of convergence of the series is unity.

(2) Consider the series

$$\Phi(z) = \sum_{1}^{\infty} \frac{2z}{z^2 - n^2\pi^2}.$$

The function $\frac{2z}{z^2 - n^2\pi^2}$ is analytic everywhere except at points $z = \pm n\pi$, where it becomes infinite. Hence, each term in the above series is an analytic function in the circle $|z| < \pi$. Let us show that the series converges uniformly inside this circle. Indeed, if $|z| \leqslant \rho$ where $\rho < \pi$, then

$$\left|\frac{2z}{z^2 - n^2\pi^2}\right| \leqslant \frac{2\rho}{n^2\pi^2 - \rho^2} = \frac{2\rho}{n^2\pi^2} \frac{1}{1-\frac{\rho^2}{n^2\pi^2}} \leqslant \frac{2\rho}{n^2\pi^2} \frac{1}{1-\frac{\rho^2}{\pi^2}},$$

and since the right-hand side is a term of a converging series $\left(\text{the series } \sum_{1}^{\infty} \frac{1}{n^2} \text{ converges}\right)$, the series $\sum_{1}^{\infty} \frac{2z}{z^2 - k^2\pi^2}$ converges uniformly inside $|z| < \pi$. Therefore, the coefficient of z^k in the Taylor expansion for $\Phi(z)$ is simply the sum of the coefficients of z^k in the Taylor expansions for each term $\frac{2z}{z^2 - n^2\pi^2}$. Such an expansion has the form

$$\frac{2z}{z^2 - n^2\pi^2} = -\frac{2z}{n^2\pi^2} \frac{1}{1-\left(\frac{z}{n\pi}\right)^2}$$

$$= -\frac{2z}{n^2\pi^2} - \frac{2z^3}{n^4\pi^4} - \frac{2z^5}{n^6\pi^6} - \ldots - \frac{2z^{2q-1}}{n^{2q}\pi^{2q}} - \ldots.$$

Here the coefficient of z^k is zero if k is even and is $-2/n^{2q}\pi^{2q}$ if $k = 2q - 1$ (odd). Hence, the coefficient of z^k in the Taylor expansion for $\Phi(z)$ is zero if k is even and $-2 \sum_{1}^{\infty} \frac{1}{(n\pi)^{2q}}$ if k is odd (equal

to $2q - 1$). Consequently,

$$\Phi(z) = -2\sum_{q=1}^{\infty}\left[\sum_{n=1}^{\infty}\frac{1}{(\pi n)^{2q}}\right]z^{2q-1}.$$

This is the sought for power series. It converges in $|z| < \pi$ since $\Phi(z)$ is analytic in this circle. At point $z = \pi$ each term in this series satisfies the inequality

$$\sum_{n=1}^{\infty}\frac{1}{(\pi n)^{2q}}\pi^{2q-1} = \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1}{n^{2q}} > \frac{1}{\pi},$$

which implies that the necessary condition for convergence of a series is not met and the series is divergent. The radius of convergence for this series, therefore, is π.

6.13

SUBSTITUTING A SERIES INTO A SERIES

Suppose that $f(z)$ can be represented in the form

$$f(z) = F[\varphi(z)], \tag{6.26}$$

where

$$\varphi(z) = \alpha_0 + \alpha_1 z + \alpha_2 z^2 + \ldots + \alpha_n z^n + \ldots \; (|z| < r) \tag{6.27}$$

and

$$F(w) = A_0 + A_1(w - \alpha_0) + A_2(w - \alpha_0)^2 + \ldots$$
$$\ldots + A_m(w - \alpha_0)^m + \ldots \tag{6.28}$$
$$(|w - \alpha_0| < R);$$

the coefficients in the power series for $\varphi(z)$ and $F(w)$ are assumed to be known. Since under our assumptions $\varphi(z) \to \alpha_0$ as $z \to 0$, there exists a ρ $(0 < \rho \leqslant r)$ such that $|\varphi(z) - \alpha_0|$ is smaller than R when $|z| < \rho$. Then the point $w = \varphi(z)$ belongs to the circle of convergence for (6.27) and, hence, the function $f(z) = F(w) = F[\varphi(z)]$ is analytic when $|z| < \rho$. This implies that there exists an expansion for $f(z)$ in a power series in z that converges for $|z| < \rho$. The problem is to calculate the expansion coefficients.

Consider the expansion

$$f(z) = F[\varphi(z)] = \sum_{0}^{\infty} A_n[\varphi(z) - \alpha_0]^n, \tag{6.29}$$

which we know to converge when $|z| < \rho$. In order to speak of uniform convergence, we substitute for ρ a number ρ' that is not

greater than ρ $(0 < \rho' \leqslant \rho)$, so that in the circle $|z| < \rho'$ the inequality

$$|\varphi(z)-\alpha_0| < \frac{R}{2}$$

is valid.

Since the series (6.28) is uniformly convergent for $|w-\alpha_0| < R/2$, the series (6.29) is uniformly convergent for $|z| < \rho'$, too. Hence, the coefficient of z^k in the Taylor expansion for $f(z)$ can be found by adding the respective coefficients in the Taylor expansions for the functions $A_n[\varphi(z)-\alpha_0]^n$.

The expansions for these functions are obtained by multiplying the expansion for $\varphi(z)-\alpha_0$ into itself n times. In our case termwise multiplication of series is valid since we are dealing with a power series in its circle of convergence, where it converges absolutely.

As a result we arrive at the following proposition: *to obtain the Taylor expansion for a function* $f(z) = F[\varphi(z)]$, *where* $\varphi(z)$ *is a function analytic in the neighborhood of the origin of coordinates and* $F(w)$ *is a function analytic in the neighborhood of the point* $\alpha_0 = \varphi(0)$, *we must substitute the expansion for* $w = \varphi(z)$ (6.27) *into the expansion for* $F(w)$ (6.28), *multiply the series, and add the coefficients of like terms. The resulting series is the sought for Taylor expansion for* $f(z)$. *It converges for certain in the circle* $|z| < \rho$, *where* ρ *is chosen such that* $|\varphi(z)-\alpha_0| < R$ *for* $|z| < \rho$.

This method of obtaining power series is called the *substitution of a series into a series.*

Examples. (1) We consider the function

$$f(z)=\sqrt{\cos z},$$

where we take the branch of this two-valued function which at $z = 0$ attains the value 1. To apply the above method we represent our function as

$$f(z) = [1-(1-\cos z)]^{1/2}.$$

Then

$$w=\varphi(z)=1-\cos z=\frac{z^2}{2!}-\frac{z^4}{4!}+\frac{z^6}{6!}-\dots \quad (|z|<\infty),$$

$$F(w)=(1-w)^{1/2}=1-\frac{1}{2}w-\frac{1}{8}w^2-\frac{1}{16}w^3-\frac{5}{128}w^4-\dots \quad (|w|<1).$$

Hence,

$$f(z)=F[\varphi(z)]=1-\frac{1}{2}\left(\frac{z^2}{2}-\frac{z^4}{24}+\frac{z^6}{720}-\dots\right)-\frac{1}{8}\left(\frac{z^2}{2}-\frac{z^4}{24}+\dots\right)^2-\frac{1}{16}\left(\frac{z^2}{2}-\dots\right)^3-\dots$$

We confine ourselves to the sixth power of z inclusive. Then we can disregard the omitted terms, since they will lead to terms with powers higher than the sixth. We have

$$\left(\frac{z^2}{2}-\frac{z^4}{24}+\ldots\right)^2=\frac{z^4}{4}-\frac{z^6}{24}+\ldots,\quad \left(\frac{z^2}{2}-\ldots\right)^3=\frac{z^6}{8}-\ldots,$$

which implies that

$$f(z)=1-\frac{1}{2}\left(\frac{z^2}{2}-\frac{z^4}{24}+\frac{z^6}{720}-\ldots\right)-\frac{1}{8}\left(\frac{z^4}{4}-\frac{z^6}{24}+\ldots\right)$$
$$-\frac{1}{16}\left(\frac{z^6}{8}-\ldots\right)+\ldots=1-\frac{z^2}{4}-\frac{z^4}{96}+\frac{19z^6}{5760}-\ldots.$$

Since $f(z)=\sqrt{\cos z}$ is a function analytic in the circle $|z|<\pi/2$ (the derivative of this function in the circle is $-\sin z/2\sqrt{\cos z}$), the series for $f(z)$ converges at $|z|<\pi/2$.

(2) Consider the function

$$f(z)=\exp\frac{1}{1-z}.$$

If we write the function $f(z)$ in the form $f(z)=e\exp\frac{z}{1-z}$, we have

$$w=\varphi(z)=\frac{z}{1-z}=z+z^2+z^3+\ldots\quad(|z|<1)$$

and

$$F(w)=e^{w+1}=e\left(1+\frac{w}{1!}+\frac{w^2}{2!}+\ldots\right)\quad(|w|<\infty).$$

Substitution of series into series yields

$$f(z)=e\left[1+(z+z^2+z^3+\ldots)\right.$$
$$\left.+\frac{(z+z^2+z^3+\ldots)^2}{2!}+\frac{(z+z^2+z^3+\ldots)^3}{3!}+\ldots\right].$$

Here $\varphi(z)=\frac{z}{1-z}$, and we can avoid multiplication of series directly by noting that for $|z|<1$

$$(z+z^2+z^3+\ldots)^k=\left(\frac{z}{1-z}\right)^k=z^k(1-z)^{-k}$$
$$=z^k\left[1+\frac{k}{1}z+\frac{k(k+1)}{2!}z^2+\ldots+\frac{k\ldots(k+n-1)}{n!}z^n+\ldots\right].$$

But since

$$\frac{k(k+1)\ldots(k+n-1)}{n!}=\frac{(k+n-1)(k+n-2)\ldots(n+1)}{(k-1)!}=\binom{k+n-1}{k-1}.$$

we have

$$(z+z^2+z^3+\ldots)^k=\sum_{n=0}^{\infty}\binom{k+n-1}{k-1}z^{n+k}.$$

Whence

$$f(z)=e\left[1+\frac{1}{1!}\sum_{0}^{\infty}z^{n+1}+\frac{1}{2!}\sum_{0}^{\infty}(n+1)z^{n+2}\right.$$

$$\left.+\frac{1}{3!}\sum_{0}^{\infty}\frac{(n+2)(n+1)}{2!}z^{n+3}+\ldots+\frac{1}{k!}\sum_{0}^{\infty}\binom{k+n-1}{k-1}z^{n+k}+\ldots\right]$$

$$=e\left\{1+z+\left(\frac{1}{1!}+\frac{1}{2!}\right)z^2+\left(\frac{1}{1!}+2\frac{1}{2!}+\frac{1}{3!}\right)z^3\right.$$

$$+\left(\frac{1}{1!}+3\frac{1}{2!}+3\frac{1}{3!}+\frac{1}{4!}\right)z^4+\ldots+\left[\frac{1}{1!}+\binom{n-1}{1}\frac{1}{2!}\right.$$

$$\left.\left.+\binom{n-1}{2}\frac{1}{3!}+\ldots+\binom{n-1}{n-2}\frac{1}{(n-1)!}+\frac{1}{n!}\right]z^n+\ldots\right\}.$$

This is the sought for expansion. The function $f(z)$ is analytic in the circle $|z|<1$; therefore, the series converges in the unit circle. Since all the expansion coefficients in the series on braces are positive numbers no smaller than unity, the series diverges at $z=1$. This implies that the radius of convergence is equal to unity.

6.14

DIVIDING POWER SERIES

Let

$$a_0+a_1(z-a)+\ldots+a_n(z-a)^n+\ldots \tag{6.30}$$

and

$$b_0+b_1(z-a)+\ldots+b_n(z-a)^n+\ldots \tag{6.31}$$

be two power series with positive radii of convergence r and ρ, where the absolute term b_0 in the second series is nonzero. We denote one of the smaller of the two numbers r and ρ by σ, i.e. $\sigma=\min(r,\rho)$ (if $r=\rho$, then $\sigma=r=\rho$). Then both series converge in the circle $|z-a|<\sigma$. If this circle contains zeros of the series (6.31), we take a new circle with a radius so small that the sum of (6.31) does not nullify inside it (this can always be done because point a is not a zero of the sum of (6.31) due to the fact that $b_0\neq 0$). Thus, there exists a circle $|z-a|<R$ inside which both series converge and the sum of the second series does not have its zeros. In this circle the quotient

$$f(z)=\frac{a_0+a_1(z-a)+\ldots+a_n(z-a)^n+\ldots}{b_0+b_1(z-a)+\ldots+b_n(z-a)^n+\ldots} \tag{6.32}$$

represents an analytic function, as follows from the rule of differentiation of a quotient. Therefore, there exists a power series

$$c_0 + c_1 (z - a) + \ldots + c_n (z - a)^n + \ldots \qquad (6.33)$$

that represents $f(z)$ inside the circle $|z - a| < R$. We call this series the *quotient* resulting from the *division* of the series (6.30) (the *dividend*) by the series (6.31) (the *divisor*).

To divide the two series we use, first, the *method of undetermined coefficients*. To this end we rewrite (6.32) in the form

$$\begin{aligned}&[c_0 + c_1 (z - a) + \ldots + c_n (z - a)^n + \ldots] \\ &\quad \times [b_0 + b_1 (z - a) + \ldots + b_n (z - a)^n + \ldots] \\ &\quad\quad = a_0 + a_1 (z - a) + \ldots + a_n (z - a)^n + \ldots\end{aligned}$$

and note that the power series, which converge inside the circle $|z - a| < R$, are also absolutely convergent. Therefore, we can multiply the two series on the right-hand side of the last equation termwise. The result is

$$\begin{aligned}&c_0b_0 + (c_0b_1 + c_1b_0)(z - a) + (c_0b_2 + c_1b_1 + c_2b_0)(z - a)^2 + \\ &\quad \ldots + (c_0b_n + c_1b_{n-1} + \ldots + c_nb_0)(z - a)^n + \ldots \\ &\quad = a_0 + a_1 (z - a) + \ldots + a_n (z - a)^n + \ldots . \qquad (6.34)\end{aligned}$$

From the fact that the sums of the left- and right-hand sides coincide inside $|z - a| < R$ it follows that, by the uniqueness property of power series, the expansion coefficients of these two series coincide. We arrive at the following system of equations:

$$\left.\begin{aligned}c_0b_0 &= a_0, \\ c_0b_1 + c_1b_0 &= a_1, \\ c_0b_2 + c_1b_1 + c_2b_0 &= a_2, \\ \ldots\ldots\ldots\ldots&\ldots\ldots \\ c_0b_n + c_1b_{n-1} + c_2b_{n-2} + \ldots + c_nb_0 &= a_n, \\ \ldots\ldots\ldots\ldots&\ldots\ldots\end{aligned}\right\} \qquad (6.35)$$

This is an infinite system of linear equations in the unknown coefficients $c_0, c_1, c_2, \ldots, c_n, \ldots$. A special feature of this set, which simplifies solution, is that for any n ($n = 0, 1, 2, 3, \ldots$) the first $n + 1$ equations contain only the first $n + 1$ unknowns. If we find c_0 from the first equation, i.e. $c_0 = a_0/b_0$ (where $b_0 \neq 0$ by assumption), and substitute it into the second equation, we arrive at the equation $(a_0/b_0) b_1 + c_1b_0 = a_1$, whence

$$c_1 = \frac{a_1b_0 - a_0b_1}{b_0^2}.$$

In general, let us suppose that we have found the first n coefficients $c_0, c_1, \dots, c_{n-1}$. Substituting them into the $(n+1)$st equation, we find that

$$c_n = \frac{a_n - c_0 b_n - c_1 b_{n-1} - \dots - c_{n-1} b_1}{b_0}. \tag{6.36}$$

In this way we can find a coefficient with any preassigned index. It is easy to find an expression for c_n in terms of the $a_0, a_1, a_2, \dots, a_n$ and $b_0, b_1, \dots, b_n$ by using determinants. The system determinant for the first $n+1$ equations is

$$\begin{vmatrix} b_0 & 0 & 0 & \dots & 0 \\ b_1 & b_0 & 0 & \dots & 0 \\ b_2 & b_1 & b_0 & \dots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ b_n & b_{n-1} & b_{n-2} & \dots & b_0 \end{vmatrix} = b_0^{n+1} \neq 0,$$

whence

$$c_n = \frac{1}{b_0^{n+1}} \begin{vmatrix} b_0 & 0 & 0 & \dots & a_0 \\ b_1 & b_0 & 0 & \dots & a_1 \\ b_2 & b_1 & b_0 & \dots & a_2 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ b_n & b_{n-1} & b_{n-2} & \dots & a_n \end{vmatrix}. \tag{6.37}$$

This formula solves the problem of series division.

Let us show that the quotient (6.33) of two power series can be found by dividing (6.30) by (6.31) in the same way as if the two series were polynomials ordered in increasing powers of $z-a$. To this end we begin to divide one by the other. This yields

$$\begin{array}{r|l}
 & \dfrac{a_0}{b_0} + \dfrac{a_1 b_0 - a_0 b_1}{b_0^2}(z-a) + \dots \\
\hline
b_0 + b_1(z-a) + \dots + b_n(z-a)^n + \dots & a_0 + a_1(z-a) + \dots + a_n(z-a)^n + \dots \\
 & a_0 + \dfrac{a_0}{b_0} b_1(z-a) + \dots + \dfrac{a_0}{b_0} b_n(z-a)^n + \dots \\
\cline{2-2}
 & \dfrac{a_1 b_0 - a_0 b_1}{b_0}(z-a) + \dots + \dfrac{a_n b_0 - a_0 b_n}{b_0}(z-a)^n + \dots \\
 & \dfrac{a_1 b_0 - a_0 b_1}{b_0}(z-a) + \dots + \dfrac{a_1 b_0 - a_0 b_1}{b_0^2} b_{n-1}(z-a)^n + \dots \\
\cline{2-2}
 & \dots\dots\dots\dots\dots\dots\dots\dots
\end{array}$$

The first two coefficients of the quotient coincide with c_0 and c_1 found via Eqs. (6.35). Suppose we have in this fashion found the first n coefficients of the quotient and they coincide with $c_0, c_1, \dots$

$\ldots, c_{n-1}$ found via Eqs. (6.35). Then we have

$$
\begin{array}{rl}
 & c_0+c_1(z-a)+\ldots+c_{n-1}(z-a)^n+\ldots \\ \hline
b_0+b_1(z-a)+\ldots+b_n(z-a)^n+\ldots \,\big) & a_0+a_1(z-a)+a_2(z-a)^2+\ldots+a_n(z-a)^n+\ldots \\
 & b_0c_0+b_1c_0(z-a)+b_2c_0(z-a)^2+\ldots+b_nc_0(z-a)^n+\ldots \\ \hline
\text{1st remainder} \left\{ \right. & (a_1-b_1c_0)(z-a)+(a_2-b_2c_0)(z-a)^2+ \\
 & \qquad \ldots+(a_n-b_nc_0)(z-a)^n+\ldots \\
 & b_0c_1(z-a)+b_1c_1(z-a)^2+\ldots+b_{n-1}c_1(z-a)^n+\ldots \\ \hline
\text{2nd remainder} \left\{ \right. & (a_2-b_2c_0-b_1c_1)(z-a)^2+ \\
 & \qquad \ldots+(a_n-b_nc_0-b_{n-1}c_1)(z-a)^n+\ldots \\
 & \ldots\ldots\ldots\ldots\ldots\ldots \\ \hline
n\text{th remainder} \left\{ \right. & (a_n-b_nc_0-b_{n-1}c_1-\ldots-b_1c_{n-1})(z-a)^n+\ldots \\
 & \ldots\ldots\ldots\ldots\ldots\ldots
\end{array}
$$

The first term in the nth remainder is $(a_n-b_nc_0-b_{n-1}c_1-\ldots \ldots-b_1c_{n-1})(z-a)^n$; therefore, the term in the quotient following $c_{n-1}(z-a)^{n-1}$ is

$$\frac{a_n-b_nc_0-b_{n-1}c_1-\ldots-b_1c_{n-1}}{b_0}(z-a)^n.$$

We see that the coefficient of this term coincides with c_n found via (6.36) from Eqs. (6.35).

Hence, *the method of undetermined coefficients when applied to division of series leads to the same result as the method in which the series are considered to be polynomials ordered in increasing powers of* $x=z-a$.

Here is an example illustrating the above reasoning. Consider the function

$$F(z)=\frac{z}{e^z-1}.$$

This is an analytic function at all points of the complex plane except the zeros of e^z-1, i.e. except points $0, \pm 2\pi i, \pm 4\pi i, \ldots$. Substituting for e^z-1 its expansion

$$e^z-1=\frac{z}{1}+\frac{z^2}{2!}+\ldots+\frac{z^n}{n!}+\ldots$$

and factoring out z from the numerator and denominator, we arrive at the following expression for $F(z)$ (which defines the function $F(z)$ at $z=0$, too):

$$F(z)=\frac{1}{1+\frac{z}{2!}+\ldots+\frac{z^n}{(n+1)!}+\ldots}.$$

The series in the denominator converges for any z and has the same zeros as the function $e^z - 1$ except one zero at the origin of coordinates. (All this follows from the fact that this series represents $(e^z - 1)/z$ for $z \neq 0$.) Therefore, in the circle $|z| < 2\pi$ its sum does not vanish and $F(z)$ can be expanded in a series in this circle by using the methods of series division. The first equation in (6.35) yields

$$c_0 \times 1 = 1, \quad \text{i.e. } c_0 = 1.$$

Since all the coefficients in the dividend series are zeros except the first, the $(n+1)$st equation in (6.35) is

$$c_0 \frac{1}{(n+1)!} + c_1 \frac{1}{n!} + \ldots + c_{n-1} \frac{1}{2!} + c_n = 0 \quad (n = 1, 2, 3, \ldots). \tag{6.38}$$

This equation enables us to determine the c's one after another. To find c_n we can also use (6.37): $c_0 = 1$,

$$c_n = \begin{vmatrix} 1 & 0 & 0 & \ldots & 1 \\ \frac{1}{2!} & 1 & 0 & \ldots & 0 \\ \frac{1}{3!} & \frac{1}{2!} & 1 & \ldots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \frac{1}{(n+1)!} & \frac{1}{n!} & \frac{1}{(n-1)!} & \ldots & 0 \end{vmatrix}$$

$$= (-1)^n \begin{vmatrix} \frac{1}{2!} & 1 & 0 & \ldots & 0 \\ \frac{1}{3!} & \frac{1}{2!} & 1 & \ldots & 0 \\ \frac{1}{4!} & \frac{1}{3!} & \frac{1}{2!} & \ldots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \frac{1}{(n+1)!} & \frac{1}{n!} & \frac{1}{(n-1)!} & \ldots & \frac{1}{2!} \end{vmatrix} \quad (n = 1, 2, 3, \ldots).$$

The quantity $c_n n!$ is known as the *nth Bernoulli number* and denoted by B_n. The coefficients of many important relationships are easily described in terms of the Bernoulli numbers. We have the

following formulas for calculating these numbers: $B_0 = c_0 \times 0! = 1$,

$$B_n = c_n n! = (-1)^n n! \begin{vmatrix} \frac{1}{2!} & 1 & 0 & \dots & 0 \\ \frac{1}{3!} & \frac{1}{2!} & 1 & \dots & 0 \\ \frac{1}{4!} & \frac{1}{3!} & \frac{1}{2!} & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ \frac{1}{(n+1)!} & \frac{1}{n!} & \frac{1}{(n-1)!} & \dots & \frac{1}{2!} \end{vmatrix}$$

$$(n = 1, 2, 3, \dots). \qquad (6.39)$$

However, an easier method of obtaining the Bernoulli numbers is to use (6.38) to compute them successively. This formula yields

$$B_0 \frac{1}{0!\,(n+1)!} + B_1 \frac{1}{1!n!} + \dots + B_n \frac{1}{n!1!} = 0 \quad (n = 1, 2, 3, \dots).$$

Multiplying both sides by $(n+1)!$ and noting that $\frac{(n+1)!}{k!\,(n+1-k)!}$ is the binomial coefficient $\binom{n+1}{k}$, we have

$$B_0 \binom{n+1}{0} + B_1 \binom{n+1}{1} + \dots + B_n \binom{n+1}{n} = 0 \quad (n = 1, 2, 3, \dots).$$

This formula can be rewritten in the symbolic form

$$(1 + B)^{n+1} - B^{n+1} = 0. \qquad (6.40)$$

where after raising $1 + B$ to the $(n + 1)$st power we must substitute indices for the exponents.

Since $B_0 = 1$, we find successively that

$$B_0 + 2B_1 = 0, \qquad B_1 = -\frac{1}{2} B_0 = -\frac{1}{2},$$

$$B_0 + 3B_1 + 3B_2 = 0, \qquad B_2 = -\frac{1}{3} B_0 - B_1 = \frac{1}{6},$$

$$B_0 + 4B_1 + 6B_2 + 4B_3 = 0, \quad B_3 = -\frac{1}{4} B_0 - B_1 - \frac{3}{2} B_2 = 0,$$

$$B_0 + 5B_1 + 10B_2 + 10B_3 + 5B_4 = 0,$$

$$B_4 = -\frac{1}{5} B_0 - B_1 - 2B_2 - 2B_3 = -\frac{1}{30},$$

$$B_0 + 6B_1 + 15B_2 + 20B_3 + 15B_4 + 6B_5 = 0,$$

$$B_5 = -\frac{1}{6} B_0 - B_1 - \frac{5}{2} B_2 - \frac{10}{3} B_3 - \frac{5}{2} B_4 = 0,$$

$$B_0 + 7B_1 + 21B_2 + 35B_3 + 35B_4 + 21B_5 + 7B_6 = 0,$$

$$B_6 = -\frac{1}{7} B_0 - B_1 - 3B_2 - 5B_3 - 5B_4 - 3B_5 = \frac{1}{42},$$

. . . .

Hence,

$$B_0 = 1, \quad B_1 = -\frac{1}{2}, \quad B_2 = \frac{1}{6}, \quad B_3 = 0,$$

$$B_4 = -\frac{1}{30}, \quad B_5 = 0, \quad B_6 = \frac{1}{42}, \; \ldots \; .$$

Let us show that *all Bernoulli numbers whose indices are odd and greater than unity are zeros*:

$$B_{2k+1} = 0 \quad (k = 1, 2, 3, \ldots).$$

To prove this proposition we substitute $-z$ for z in the expansion

$$\frac{z}{e^z - 1} = c_0 + c_1 z + c_2 z^2 + \ldots + c_n z^n + \ldots$$

$$= B_0 + \frac{B_1}{1!} z + \frac{B_2}{2!} z^2 + \ldots + \frac{B_n}{n!} z^n + \ldots . \tag{6.41}$$

This yields

$$\frac{-z}{e^{-z} - 1} = -\frac{ze^z}{(e^{-z} - 1) e^z} = \frac{ze^z}{e^z - 1} = B_0 - \frac{B_1}{1!} z + \frac{B_2}{2!} z^2 - \frac{B_3}{3!} z^3 + \ldots,$$

or, after we have subtracted this expansion from (6.41),

$$\frac{z}{e^z - 1} - \frac{ze^z}{e^z - 1} = -z = 2\frac{B_1}{1!} z + 2\frac{B_3}{3!} z^3 + \ldots$$

$$+ 2\frac{B_{2k+1}}{(2k+1)!} z^{2k+1} + \ldots .$$

Then the uniqueness of an expansion in a power series yields

$$2B_1 = -1, \quad B_3 = B_5 = \ldots = B_{2k+1} = \ldots = 0,$$

which is what we set out to prove.

If we use this property of Bernoulli numbers, we can write expansion (6.41) in the form

$$\frac{z}{e^z - 1} = 1 - \frac{z}{2} + \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!} z^{2k}. \tag{6.42}$$

6.15

POWER SERIES EXPANSIONS FOR COT Z, TAN Z, CSC Z, AND SEC Z

Expansion (6.42) can be used to find the power series for $z \cot z$, $\tan z$, and $z \csc z$. We write

$$\cot z = \frac{\cos z}{\sin z} = i\frac{e^{iz} + e^{-iz}}{e^{iz} - e^{-iz}} = i\frac{e^{2iz} + 1}{e^{2iz} - 1} = i + \frac{2i}{e^{2iz} - 1},$$

whence

$$z \cot z = iz + \frac{2iz}{e^{2iz} - 1}.$$

We can use (6.42) to expand the function $2iz/(e^{2iz}-1)$ if we substitute $2iz$ for z in the former. Since the series (6.42) converges for $|z|<2\pi$, the sought for series converges for $|2iz|<2\pi$, i.e. for $|z|<\pi$. Hence,

$$\frac{2iz}{e^{2iz}-1}=1-\frac{2iz}{2}+\sum_{k=1}^{\infty}\frac{B_{2k}}{(2k)!}(2iz)^{2k}=1-iz+\sum_{k=1}^{\infty}(-1)^k\frac{2^{2k}B_{2k}}{(2k)!}z^{2k}$$

and, therefore,

$$z\cot z=1+\sum_{k=1}^{\infty}(-1)^k\frac{2^{2k}B_{2k}z^{2k}}{(2k)!}. \tag{6.43}$$

To find the Taylor expansion for $\tan z$, we note that

$$\cot z-\tan z=2\cot 2z,$$

whence

$$\tan z=\cot z-2\cot 2z.$$

Substituting $2z$ for z in (6.43), we arrive at a series that converges for $|2z|<\pi$, i.e. for $|z|<\pi/2$:

$$2z\cot 2z=1+\sum_{k=1}^{\infty}(-1)^k\frac{2^{4k}B_{2k}}{(2k)!}z^{2k}. \tag{6.44}$$

Subtracting (6.44) from (6.43) termwise, we find that

$$z\cot z-2z\cot 2z=z\tan z=\sum_{k=1}^{\infty}(-1)^k\frac{2^{2k}(1-2^{2k})B_{2k}}{(2k)!}z^{2k},$$

or

$$\tan z=\sum_{k=1}^{\infty}(-1)^{k-1}\frac{2^{2k}(2^{2k}-1)B_{2k}}{(2k)!}z^{2k-1}. \tag{6.45}$$

It is obvious then that by construction this series converges if $|z|$ is less than $\pi/2$.

Turning to the function $z\csc z$, we note that

$$\cot z+\tan\frac{z}{2}=\frac{\cos z\cos\frac{z}{2}+\sin z\sin\frac{z}{2}}{\sin z\cos\frac{z}{2}}=\frac{\cos\frac{z}{2}}{\sin z\cos\frac{z}{2}}=\csc z.$$

Substituting $z/2$ for z in (6.45), we arrive at a series that converges for $|z/2|<\pi/2$, i.e. for $|z|<\pi$. From this and from series (6.43), which also converges for $|z|<\pi$, we find that

$$z\csc z=z\cot z+z\tan\frac{z}{2}=1+\sum_{k=1}^{\infty}(-1)^{k-1}\frac{(2^{2k}-2)B_{2k}}{(2k)!}z^{2k}. \tag{6.46}$$

Finally, let us find the expansion formula for sec z. Since this function is analytic in the circle $|z| < \pi/2$, the sought for expansion converges in the same circle. Once more we employ division of series. This yields

$$\sec z = \frac{1}{\cos z} = \frac{1}{1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \ldots + (-1)^k \frac{z^{2k}}{(2k)!} + \ldots}$$
$$= a_0 + a_1 z + a_2 z^2 + \ldots .$$

From the fact that sec z is an even function it follows that the coefficients of odd powers of z, i.e. $a_1, a_3, a_5, \ldots$, are zeros:

$$\sec z = \frac{1}{1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \ldots} = a_0 + a_2 z^2 + a_4 z^4 + \ldots . \tag{6.47}$$

The expansion coefficients with even indices, $a_0, a_2, a_4, \ldots$, are usually written in the form

$$a_{2k} = (-1)^k \frac{E_{2k}}{(2k)!} \quad (k = 0, 1, 2, \ldots),$$

where the quantities E_{2k} are known as the *Euler numbers*. If we write (6.47) as

$$1 = \left(1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \ldots\right)\left(E_0 - \frac{E_2}{2!} z^2 + \frac{E_4}{4!} z^4 - \ldots\right)$$

and multiply the series on the right-hand side, we find that

$$1 = E_0 - \left(\frac{E_0}{2!} + \frac{E_2}{2!}\right) z^2$$
$$+ \left(\frac{E_0}{4!} + \frac{E_2}{2!\,2!} + \frac{E_4}{4!}\right) z^4 - \left(\frac{E_0}{6!} + \frac{E_2}{4!\,2!} + \frac{E_4}{2!\,4!} + \frac{E_6}{6!}\right) z^6 + \ldots,$$

whence

$$E_0 = 1,$$
$$\frac{E_0}{2!} + \frac{E_2}{2!} = 0,$$
$$\frac{E_0}{4!} + \frac{E_2}{2!\,2!} + \frac{E_4}{4!} = 0,$$
$$\frac{E_0}{6!} + \frac{E_2}{4!\,2!} + \frac{E_4}{2!\,4!} + \frac{E_6}{6!} = 0,$$
$$\cdots\cdots\cdots\cdots\cdots$$

These equations enable us to find the Euler numbers successively. We have

$$E_0 = 1, \quad E_2 = -1, \quad E_4 = 5, \quad E_6 = -61, \quad E_8 = 1385, \quad \ldots .$$

In general, if we have found $E_0, E_2, \ldots, E_{2n-2}$, the number E_{2n} is determined from the equation

$$\frac{E_0}{(2n)!}+\frac{E_2}{(2n-2)!\,2!}+\frac{E_4}{(2n-4)!\,4!}+\ldots+\frac{E_{2n}}{(2n)!}=0,$$

or

$$E_0+\binom{2n}{2}E_2+\binom{2n}{4}E_4+\ldots+\binom{2n}{2n-2}E_{2n-2}+E_{2n}=0.$$

This implies that if the numbers $E_0, E_2, \ldots, E_{2n-2}$ are integers, then so is E_{2n}. But since the first Euler numbers are integers, all Euler numbers are integers, too.

We can finally write the expansion for sec z in the following form:

$$\sec z=\sum_0^\infty(-1)^k\frac{E_{2k}}{(2k)!}z^{2k}. \tag{6.48}$$

It converges for $|z|<\pi/2$. The Euler numbers, which enter the expansion coefficients, are fully determined by the condition that

$$E_0=1,\ \ E_0+\binom{2n}{2}E_2+\binom{2n}{4}E_4+\ldots+\binom{2n}{2n-2}E_{2n-2}+E_{2n}=0$$

$$(n=1, 2, 3, \ldots). \tag{6.49}$$

6.16

SERIES EXPANSIONS FOR HARMONIC FUNCTIONS. THE POISSON INTEGRAL AND SCHWARZ'S FORMULA

By using the fact that harmonic functions are closely related to analytic functions we can derive the basic properties of the former from the already known properties of the latter.

Suppose that $u(x, y)$ is a single-valued function harmonic in a certain circle K: $|z-z_0|<R$. Then according to Sec. 2.13 there exists a single-valued harmonic function $v(x, y)$ in K conjugate to $u(x, y)$. Let us build the corresponding analytic function $f(z) = u(x, y)+iv(x, y)$ and expand it in a power series in $z-z_0$. We have

$$f(z)=\sum_0^\infty(\alpha_n+i\beta_n)(z-z_0)^n, \tag{6.50}$$

where α_n and β_n are real numbers. Next we introduce the polar coordinates r and θ with the pole at point z_0, so that $z = z_0 + re^{i\theta}$, and use the notations $u(r, \theta)$ and $v(r, \theta)$ as equivalent to $u(x, y)$ and $v(x, y)$. Separating in (6.50) the real and imaginary parts, we have two series:

$$u(r, \theta) = \alpha_0 + \sum_{1}^{\infty} (\alpha_n \cos n\theta - \beta_n \sin n\theta) r^n, \tag{6.51}$$

$$v(r, \theta) = \beta_0 + \sum_{0}^{\infty} (\beta_n \cos n\theta + \alpha_n \sin n\theta) r^n, \tag{6.52}$$

which converge uniformly inside K.

Therefore, *every function* $u(r, \theta)$ *that is harmonic in a circle* $|z - z_0| < R$ *admits in the circle an expansion of the type* (6.51) *that converges uniformly inside the circle.* The expansion coefficients are real numbers α_n and $-\beta_n$. Since the power series (6.50) converges in the given circle and perhaps has a radius of convergence greater than R, these numbers must satisfy the inequality

$$R \leqslant \frac{1}{\overline{\lim\limits_{n\to\infty}} \sqrt[n]{|\alpha_n + i\beta_n|}}.$$

We can easily see that if α_n and β_n are real numbers such that they satisfy this inequality, series (6.51) defines a function that is harmonic in K. To convince oneself, it suffices to note that if the above inequality is satisfied, the power series (6.50) converges in K and, hence, represents in it an analytic function whose real part is represented by the series (6.51). Thus, *if there exists an expansion of the type* (6.51) *or* (6.52) *and an appropriate inequality for the expansion coefficients, the series represents a function that is harmonic in the given circle.*

We apply the above reasoning to the function $f(z) = \frac{\rho e^{i\alpha} + (z - z_0)}{\rho e^{i\alpha} - (z - z_0)}$, whose expansion in the circle $|z - z_0| < \rho$ has the following form:

$$\frac{\rho e^{i\alpha} + (z - z_0)}{\rho e^{i\alpha} - (z - z_0)} = -1 + \frac{2\rho e^{i\alpha}}{\rho e^{i\alpha} - (z - z_0)}$$

$$= -1 + 2\left[1 + \frac{z - z_0}{\rho e^{i\alpha}} + \frac{(z - z_0)^2}{\rho^2 e^{2i\alpha}} + \dots\right] = 1 + 2\sum_{1}^{\infty} \frac{(z - z_0)^n}{\rho^n} e^{-in\alpha}.$$

This yields*

$$\operatorname{Re} f(z) = \frac{\rho^2 - r^2}{\rho^2 + r^2 - 2r\rho\cos(\theta-\alpha)} = 1 + 2\sum_1^\infty \left(\frac{r}{\rho}\right)^n \cos n(\theta-\alpha), \tag{6.53}$$

$$\operatorname{Im} f(z) = \frac{2r\rho\sin(\theta-\alpha)}{\rho^2 + r^2 - 2r\rho\cos(\theta-\alpha)} = 2\sum_1^\infty \left(\frac{r}{\rho}\right)^n \sin n(\theta-\alpha), \tag{6.54}$$

where both series converge uniformly inside $|z - z_0| < \rho$.

Now we return to the series (6.51) and (6.52). In the first we substitute an arbitrary ρ ($\rho < R$) for r, an α for θ, multiply both sides into $\cos m\alpha$, and integrate (for a fixed ρ) with respect to α from zero to 2π. This yields

$$\int_0^{2\pi} u(\rho, \alpha)\cos m\alpha\, d\alpha = \alpha_m \rho^m \int_0^{2\pi} \cos^2 m\alpha\, d\alpha,$$

whence

$$\alpha_0 = \frac{1}{2\pi}\int_0^{2\pi} u(\rho, \alpha)\, d\alpha, \quad \alpha_m = \frac{1}{\pi\rho^m}\int_0^{2\pi} u(\rho, \alpha)\cos m\alpha\, d\alpha \ (m \geqslant 1). \tag{6.55}$$

If we multiply not into $\cos m\alpha$ but into $\sin m\alpha$ and integrate from zero to 2π, we obtain

$$-\beta_m = \frac{1}{\pi\rho^m}\int_0^{2\pi} u(\rho, \alpha)\sin m\alpha\, d\alpha \ (m \geqslant 1). \tag{6.56}$$

Substituting the found expressions for α_n and β_n into (6.51) and (6.52), we obtain

$$u(r, \theta) = \frac{1}{2\pi}\int_0^{2\pi} u(\rho, \alpha)\, d\alpha + \sum_1^\infty \frac{1}{\pi}\int_0^{2\pi} u(\rho, \alpha)\cos m(\theta-\alpha)\, d\alpha \left(\frac{r}{\rho}\right)^m,$$

$$v(r, \theta) = \beta_0 + \sum_1^\infty \frac{1}{\pi}\int_0^{2\pi} u(\rho, \alpha)\sin m(\theta-\alpha)\, d\alpha \left(\frac{r}{\rho}\right)^m.$$

Suppose that ρ satisfies the condition $r < \rho < R$. We multiply (6.53) and (6.54) into $(1/2\pi)\, u(\rho, \alpha)$ and then integrate termwise with respect to α from zero to 2π (for fixed r and ρ). All this is justified

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* The real and imaginary parts of $f(z)$ can be found by multiplying the numerator and denominator into an expression that is the conjugate complex of the denominator:

$$\frac{\rho e^{i\alpha} + re^{i\theta}}{\rho e^{i\alpha} - re^{i\theta}} = \frac{(\rho e^{i\alpha} + re^{i\theta})(\rho e^{-i\alpha} - re^{-i\theta})}{(\rho e^{i\alpha} - re^{i\theta})(\rho e^{-i\alpha} - re^{-i\theta})} = \frac{\rho^2 - r^2 + i2r\rho\sin(\theta-\alpha)}{\rho^2 + r^2 - 2r\rho\cos(\theta-\alpha)}.$$

due to the uniform convergence of the series (6.53) and (6.54) inside the circle $|z - z_0| < \rho$.

Comparing these expansions with those for $u(r, \theta)$ and $v(r, \theta)$, we find that

$$u(r, \theta) = \frac{1}{2\pi} \int_0^{2\pi} u(\rho, \alpha)\left[1 + 2\sum_1^{\infty}\left(\frac{r}{\rho}\right)^n \sin n(\theta - \alpha)\right] d\alpha$$

$$= \frac{1}{2\pi} \int_0^{2\pi} u(\rho, \alpha) \frac{\rho^2 - r^2}{\rho^2 + r^2 - 2\rho r \cos(\theta - \alpha)} d\alpha, \tag{6.57}$$

$$v(r, \theta) = \beta_0 + \frac{1}{2\pi} \int_0^{2\pi} u(\rho, \alpha)\left[2\sum_1^{\infty}\left(\frac{r}{\rho}\right)^n \sin n(\theta - \alpha)\right] d\alpha$$

$$= \beta_0 + \frac{1}{2\pi} \int_0^{2\pi} u(\rho, \alpha) \frac{2r\rho \sin(\theta - \alpha)}{\rho^2 + r^2 - 2r\rho \cos(\theta - \alpha)} d\alpha. \tag{6.58}$$

Thus, for the function $u(r, \theta)$ and the conjugate function $v(r, \theta)$ we have integral representations in terms of the values of $u(\rho, \alpha)$ on the circle $|z - z_0| = \rho < R$.

The difference of the integrals in (6.57) and (6.58) is due to the fact that in the second the harmonic function $v(r, \theta)$ is expressed not in terms of the values of this function on the circle $|z - z_0| = \rho$ but in terms of the values of the conjugate function (on the same circle). But since formula (6.57) is valid for any function that is harmonic in a given circle, we have a similar formula for $v(r, \theta)$:

$$v(r, \theta) = \frac{1}{2\pi} \int_0^{2\pi} v(\rho, \alpha) \frac{\rho^2 - r^2}{\rho^2 + r^2 - 2\rho r \cos(\theta - \alpha)} d\alpha. \tag{6.59}$$

The integral on the right-hand side of (6.57) or (6.59) is known as the *Poisson integral of the function* $u(\rho, \alpha)$ or $v(\rho, \alpha)$, respectively, the formula (6.57) or (6.59) the *Poisson integral formula*, and the harmonic function

$$\frac{\rho^2 - r^2}{\rho^2 + r^2 - 2\rho r \cos(\theta - \alpha)} = \operatorname{Re}\left[\frac{\rho e^{i\alpha} + (z - z_0)}{\rho e^{i\alpha} - (z - z_0)}\right]$$

the *Poisson kernel*.

If in (6.57) we put $u(r, \theta) \equiv 1$, then

$$1 = \frac{1}{2\pi} \int_0^{2\pi} \frac{\rho^2 - r^2}{\rho^2 + r^2 - 2\rho r \cos(\theta - \alpha)} d\alpha. \tag{6.60}$$

In general, if $\varphi(\alpha)$ is a real-valued function defined and continuous in the segment $[0, 2\pi]$, we will call the integral

$$\frac{1}{2\pi}\int_0^{2\pi} \varphi(\alpha)\frac{\rho^2-r^2}{\rho^2+r^2-2\rho r\cos(\theta-\alpha)}\,d\alpha, \tag{6.61}$$

a Poisson integral and not require that $\varphi(\alpha)$ coincide with the values of a harmonic function $u(\rho, \alpha)$. It can be proved that (6.61) represents a function that is harmonic in the circle $|z - z_0| < \rho$ and that tends to $\varphi(\alpha)$ as the point (r, θ) tends to a point (ρ, α) on the circle*.

The Poisson integral is similar to Cauchy's integral along a circle and may be obtained from the latter by a certain transformation. To this end, in addition to Cauchy's integral formula

$$f(z)=\frac{1}{2\pi i}\int_{|\zeta-z_0|=\rho}\frac{f(\zeta)}{\zeta-z}\,d\zeta, \tag{6.62}$$

where point z lies in the interior of the circle $|\zeta - z_0| = \rho_0$, we will consider Cauchy's integral obtained by substituting for point z a point $z^* = z_0 + \rho^2/(\bar{z} - \bar{z}_0)$ symmetric to z with respect to the circle $|\zeta - z_0| = \rho$. Since point z^* lies in the exterior of this circle, the integral is zero:

$$0=\frac{1}{2\pi i}\int_{|\zeta-z_0|=\rho}\frac{f(\zeta)}{\xi-z^*}\,d\zeta. \tag{6.63}$$

We subtract (6.63) from (6.62) termwise and use the following notations:

$$\zeta - z = \zeta - z_0 - (z - z_0) = \rho e^{i\alpha} - re^{i\theta},$$

$$\zeta - z^* = \zeta - z_0 - (z^* - z_0) = \rho e^{i\alpha} - \frac{\rho^2}{r}e^{i\theta}, \text{ and } d\zeta = i\rho e^{i\alpha}\,d\alpha.$$

This yields

$$f(z)=\frac{1}{2\pi i}\int_{|\zeta-z_0|=\rho} f(\zeta)\left(\frac{1}{\zeta-z}-\frac{1}{\zeta-z^*}\right)d\zeta$$

$$=\frac{1}{2\pi}\int_0^{2\pi} f(\zeta)\left[\frac{\rho}{\rho-re^{i(\theta-\alpha)}}+\frac{re^{i(\alpha-\theta)}}{\rho-re^{i(\alpha-\theta)}}\right]d\alpha$$

$$=\frac{1}{2\pi}\int_0^{2\pi} f(\zeta)\frac{\rho^2-r^2}{\rho^2+r^2-2r\rho\cos(\alpha-\theta)}\,d\alpha.$$

Substituting $u(r, \theta) + iv(r, \theta)$ for $f(z)$ and $u(\rho, \alpha) + iv(\rho, \alpha)$ for $f(\zeta)$ and separating the real and imaginary parts, we arrive again at (6.57) and (6.59).

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* See, for instance, A. I. Markushevich, *The Theory of Analytic Functions* [in Russian], vol. 2, Nauka, Moscow, 1968, Chap. 6, Sec. 1.5.

From formula (6.57) and (6.58) we can easily arrive at an important formula that relates an analytic function $f(z)$ to the values of its real part on a circle. Namely, if we multiply (6.58) into i and add the result to (6.57), we obtain

$$f(z)=i\beta_0+\frac{1}{2\pi}\int_0^{2\pi} u(\rho,\alpha)\left[\frac{\rho^2-r^2}{\rho^2+r^2-2r\rho\cos(\theta-\alpha)}\right.$$
$$\left.+i\frac{2r\rho\sin(\theta-\alpha)}{\rho^2+r^2-2r\rho\cos(\theta-\alpha)}\right]d\alpha.$$

But the expression in the brackets is an analytic function of z, namely

$$\frac{\rho e^{i\alpha}+(z-z_0)}{\rho e^{i\alpha}-(z-z_0)}.$$

Therefore,

$$f(z)=i\beta_0+\frac{1}{2\pi}\int_0^{2\pi} u(\rho,\alpha)\frac{\rho e^{i\alpha}+(z-z_0)}{\rho e^{i\alpha}-(z-z_0)}\,d\alpha. \tag{6.64}$$

Here $i\beta_0$ is a pure imaginary constant representing the imaginary part of $f(z_0)$; this constant, of course, cannot be determined from the real part of $f(z)$. Formula (6.64) is known as *Schwarz's formula.*

From (6.57) we find that at $r=0$, i.e. at the center of circle K,

$$u(x_0, y_0)=\frac{1}{2\pi}\int_0^{2\pi} u(\rho,\alpha)\,d\alpha,$$

where $u(\rho,\alpha)$ denotes, as agreed, the value of $u(x, y)$ at the points on the circle $|z-z_0|=\rho$ centered at $z_0=x_0+iy_0$. A more detailed notation is

$$u(x_0, y_0)=\frac{1}{2\pi}\int_0^{2\pi} u(x_0+\rho\cos\alpha,\ y_0+\rho\sin\alpha)\,d\alpha. \tag{6.65}$$

Hence, *the value of a harmonic function in the center of a circle is the arithmetic mean of its values on a circle centered at this point.* (We discussed this fact at the end of Sec. 6.1.)

It can be proved that this property is a characteristic feature of harmonic functions. More precisely, the following proposition is valid:

*Suppose $u(x, y)$ is a real-valued function that is single-valued and analytic in a domain G. If to every point $z_0=x_0+iy_0\in G$ there corresponds a neighborhood $|z-z_0|<\delta(z_0)$ in which $u(x_0, y_0)$ is the arithmetic mean of its values on every circle $|z-z_0|=\rho$ $(0<\rho<\delta(z_0))$, then $u(x, y)$ is harmonic in G**.

* See, for example, A. I. Markushevich, *The Theory of Analytic Functions* [in Russian], vol. 2, Nauka, Moscow, 1968, Chap. 6, Sec. 3.1.

6.17
ANALYTIC FUNCTIONS OF SEVERAL COMPLEX VARIABLES

Let us consider a $2n$-dimensional Euclidean space E_{2n}. Each point of this space is represented by an ordered set of $2n$ real numbers, $x_1, y_1, x_2, y_2, \ldots, x_{2n}, y_{2n}$. We may also say that every point is represented by an ordered set of n complex numbers $z_j = x_j + iy_j$, which are the *complex coordinates* of this point. Then the space is an n-dimensional complex space denoted by $\mathbb{C}^n$. With $n = 1$ we get the complex plane; in what follows we will consider n to be greater than unity. If we want to isolate a z_k plane in $\mathbb{C}^n$, we put $z_j = 0$ for $j \neq k$, where z_k is any complex number. Such a plane can be called a *coordinate plane*; note that in an n-dimensional complex space it plays the role of a coordinate axis.

The *distance* between any two points $\{z_j'\}$ and $\{z_j''\}$ is defined thus:

$$\sqrt{\sum_{j=1}^{n} |z_j' - z_j''|^2}.$$

A neighborhood of a point $\{z_j^{(0)}\}$ can be defined as the interior of a ball with its center at this point and any positive radius ρ:

$$\sum_1^n |z_j - z_j^{(0)}|^2 < \rho^2.$$

More often, however, it is convenient to define a neighborhood as a topological product of n circles with centers at points $z_j^{(0)}$ $(j = 1, 2, \ldots, n)$. Such a neighborhood is characterized by n inequalities

$$|z_j - z_j^{(0)}| < \rho_j, \quad j = 1, \ldots, n.$$

The following concepts are introduced in a space $\mathbb{C}^n$ quite naturally: a limit point of a subset of points, a closed set, an interior point of a set, an open set, its boundaries, and a domain. The last is defined as an open and connected set.

Examples of domains. (a) A *ball* with the center at the origin of coordinates: $\sum_1^n |z_j|^2 < \rho^2$. Every z_k-plane intersects the ball along the circle $|z_k| < \rho$. The boundary of the ball is defined by the equation $\sum_1^n |z_j|^2 = \rho^2$. The circumferences $|z_k| = \rho$, $k = 1, 2, \ldots, n$, lying in the coordinate planes belong to the boundary but do not exhaust it. For instance, all points of a product of n circumferences lying in the same planes, $|z_k| = \rho/\sqrt{n}$, $k = 1, 2, \ldots, n$, constitute the boundary of the ball, too.

(b) The *product* of n circles

$$|z_j| < \rho, \quad j = 1, \ldots, n.$$

Here again each of the z_k-planes intersects the domain along the circle $|z_k| < \rho$. The boundary of the domain is the union of n sets:

$$\bigcup_{k=1}^{n} \{|z_k| = \rho, \quad |z_j| \leqslant \rho \text{ for } j \neq k\}.$$

In particular, all points of the product of the n circumferences $|z_k| = \rho$, $z_j = 0$ for $j \neq k$, $k = 1, 2, \ldots, n$, are boundary points.

The ball $\sum_{1}^{n} |z_j|^2 < \rho^2$ is an open subset of this domain. The boundaries of these two domains intersect along n circles

$$\{|z_k| = \rho, \quad z_j = 0 \quad \text{for} \quad j \neq k\}, \quad k = 1, \ldots, n.$$

In turn, the ball $\sum_{1}^{n} |z_j|^2 < \rho^2$ contains the product of n circles $|z_j| < \rho/\sqrt{n}$; the boundaries of these domains intersect along the product of n circumferences

$$\left\{|z_k| = \frac{\rho}{\sqrt{n}}, \quad z_j = 0 \text{ for } j \neq k\right\}, \quad k = 1, \ldots, n.$$

(c) A generalization of the previous examples is the *product G of n domains G_k*, each of which belongs to a plane. The boundary of G is the union of n sets, each of which is the product of the boundary of any of the G_k *and* the closure of the other $n - 1$ domains.

The part of the boundary represented by the product of boundaries of the G_k is called the *skeleton* (or *determining set*) of G.

Consider an arbitrary domain $G \subset \mathbb{C}^n$ and a function of n complex variables $f(z_1, \ldots, z_n)$ that is defined and continuous in G (a complex-valued function). This function is *analytic* in G if at each point $\{z_j^{(0)}\} \in G$ there exist the partial derivatives

$$\frac{\partial f}{\partial z_k} = \lim_{z_k \to z_k^{(0)}} \frac{f(z_1^{(0)}, \ldots, z_{k-1}^{(0)}, z_k, z_{k+1}^{(0)}, \ldots, z_n^{(0)}) - f(z_1^{(0)}, \ldots, z_k^{(0)}, \ldots, z_n^{(0)})}{z_k - z_k^{(0)}}.$$

This definition implies that at a fixed z_j, $j \neq k$, the function is analytic with respect to z_k ($k = 1, \ldots, n$). The American analyst William Fogg Osgood proved that in the above definition the requirement that the function be continuous can be dropped (in other words, if the partial derivatives exist at each point of the domain, the function is necessarily continuous).

As a closed neighborhood of point $\{z_j^{(0)}\}$ belonging to G we take the product of n closed circles

$$\{|z_k - z_k^{(0)}| \leqslant \rho_k, \quad z_j = 0 \quad \text{for} \quad j \neq k\}, \quad k = 1, \ldots, n.$$

Let us show that at each interior point of this neighborhood the value of the function may be expressed in terms of its values on the skeleton Γ of G by means of a formula similar to Cauchy's integral formula, namely

$$f(z_1, \ldots, z_n) = \frac{1}{(2\pi i)^n} \int\limits_{|\zeta_n - z_n^{(0)}| = \rho_n} d\zeta_n \int\limits_{|\zeta_{n-1} - z_{n-1}^{(0)}| = \rho_{n-1}} d\zeta_{n-1} \ldots \int\limits_{|\zeta_1 - z_1^{(0)}| = \rho_1} \frac{f(\zeta_1, \ldots, \zeta_n)}{(\zeta_1 - z_1) \ldots (\zeta_n - z_n)} d\zeta_1 \quad (6.66)$$

(point $\{z_j\}$ lies inside the neighborhood).

Indeed, according to Cauchy's integral formula,

$$\int\limits_{\zeta_1 - z_1^{(0)}| = \rho_1} \frac{f(\zeta_1, \ldots, \zeta_n)}{(\zeta_1 - z_1) \ldots (\zeta_n - z_n)} d\zeta_1 = \frac{2\pi i f(z_1, \zeta_2, \ldots, \zeta_n)}{(\zeta_2 - z_2) \ldots (\zeta_n - z_n)} \quad (|z - z_1^{(0)}| < \rho_1),$$

$$2\pi i \int\limits_{|\zeta_2 - z_2^{(0)}| = \rho_2} \frac{f(z_1, \zeta_2, \ldots, \zeta_n)}{(\zeta_2 - z_2) \ldots (\zeta_n - z_n)} d\zeta_2 = \frac{(2\pi i)^2 f(z_1, z_2, \zeta_3, \ldots, \zeta_n)}{(\zeta_3 - z_3) \ldots (\zeta_n - z_n)} \quad (|z_2 - z_2^{(0)}| < \rho_2),$$

etc. After applying Cauchy's integral formula n times we arrive at (6.66). Since this formula is valid for any order of integration, we can use a simpler notation:

$$f(z_1, \ldots, z_n) = \frac{1}{(2\pi i)^n} \int\limits_{\Gamma} \frac{f(\zeta_1, \ldots, \zeta_n)\, d\zeta_1 \ldots d\zeta_n}{(\zeta_1 - z_1^{(0)}) \ldots (\zeta_n - z_n^{(0)})}.$$

We can now use this result to establish whether an analytic function of several complex variables can be expanded in a power series in a neighborhood of each point of G. By analogy with the case where $n = 1$, it suffices to expand the fraction in the integrand in (6.66) into a power series. This yields

$$\frac{1}{(\zeta_1 - z_1) \ldots (\zeta_n - z_n)} = \frac{1}{(\zeta_1 - z_1^{(0)}) \ldots (\zeta_n - z_n^{(0)})} \frac{1}{\left(1 - \frac{z_1 - z_1^{(0)}}{\zeta_1 - z_1^{(0)}}\right) \ldots \left(1 - \frac{z_n - z_n^{(0)}}{\zeta_n - z_n^{(0)}}\right)} = \prod_{p=1}^{n} \sum_{j=0}^{\infty} \frac{(z_p - z_p^{(0)})^j}{(\zeta_p - z_p^{(0)})^{j+1}}.$$

Obviously, each of these series converges absolutely and uniformly with respect to $\{\zeta_k\}$ on the skeleton Γ of the neighborhood under

consideration if point $\{z_k\}$ is a fixed interior point of this neighborhood. Substituting into (6.66) and performing the successive integrations, we find that

$$f(z_1, \ldots, z_n) = \frac{1}{(2\pi i)^{n-1}} \int\limits_{|\zeta_n - z_n^{(0)}| = \rho_n} d\zeta_n \ldots \int\limits_{|\zeta_2 - z_2^{(0)}| = \rho_2} d\zeta_2 \Bigg[\sum_{\alpha_1=0}^{\infty} \frac{1}{2\pi i}$$

$$\times (z_1 - z_1^{(0)})^{\alpha_1} \int\limits_{|\zeta_1 - z_1^{(0)}| = \rho_1} \frac{f(\zeta_1, \ldots, \zeta_n)}{(\zeta_1 - z_1^{(0)})^{\alpha_1+1}} d\zeta_1 \prod_{p=2}^{n} \sum_{j=0}^{\infty} \frac{(z_p - z_p^{(0)})^j}{(\zeta_p - z_p^{(0)})^{j+1}} \Bigg]$$

$$= \ldots = \sum_{\alpha_1=0}^{\infty} \ldots \sum_{\alpha_n=0}^{\infty} \frac{1}{(2\pi i)^n} \int\limits_{\Gamma} \frac{f(\zeta_1, \ldots, \zeta_n)}{(\zeta_1 - z_1^{(0)})^{\alpha_1+1} \ldots (\zeta_n - z_n^{(0)})^{\alpha_n+1}}$$

$$\times d\zeta_1 \ldots d\zeta_n (z_1 - z_1^{(0)})^{\alpha_1} \ldots (z_n - z_n^{(0)})^{\alpha_n}.$$

By introducing the notations

$$\frac{1}{2\pi i} \int\limits_{\Gamma} \frac{f(\zeta_1, \ldots, \zeta_n)}{(\zeta_1 - z_1^{(0)})^{\alpha_1+1} \ldots (\zeta_n - z_n^{(0)})^{\alpha_n+1}} d\zeta_1 \ldots d\zeta_n = c_{\alpha_1\alpha_2 \ldots \alpha_n}$$

we arrive at the sought for power series

$$f(z_1, \ldots, z_n) = \sum_{(\alpha_1, \alpha_2, \ldots, \alpha_n)} \sum \ldots \sum c_{\alpha_1\alpha_2\ldots\alpha_n} (z_1 - z_1^{(0)})^{\alpha_1} \ldots (z_n - z_n^{(0)})^{\alpha_n}.$$

It is easy to establish that each series of this type that converges in a neighborhood of point $\{z_k^{(0)}\}$ (the expansion coefficients are any complex numbers that ensure convergence) represents (in this neighborhood) a continuous function differentiable in all its variables, i.e. an analytic function. The partial derivatives are obtained by termwise differentiation of the series.

This implies that

$$\alpha_1! \ldots \alpha_n! \, c_{\alpha_1\ldots\alpha_n} = \left. \frac{\partial^{\alpha_1+\ldots+\alpha_n} f(z_1, \ldots, z_n)}{\partial z_1^{\alpha_1} \ldots \partial z_n^{\alpha_n}} \right|_{z_1^{(0)}, \ldots, z_n^{(0)}}$$

and hence,

$$\left. \frac{\partial^{\alpha_1+\ldots+\alpha_n} f}{\partial z_1^{\alpha_1} \ldots \partial z_n^{\alpha_n}} \right|_{z_1^{(0)}, \ldots, z_n^{(0)}}$$

$$= \frac{\alpha_1! \ldots \alpha_n!}{(2\pi i)^n} \int\limits_{\Gamma} \frac{f(\zeta_1, \ldots, \zeta_n)}{(z_1 - z_1^{(0)})^{\alpha_1+1} \ldots (z_n - z_n^{(0)})^{\alpha_n+1}} d\zeta_1 \ldots d\zeta_n.$$

Note that the above reasoning also implies that we can define an analytic function of n complex variables in a domain G as a complex-valued point function that in a neighborhood of each point $\{z_k^{(0)}\}$

in G can be expanded in a power series of the type

$$f(z_1, \ldots, z_n) = \sum_{(\alpha_1, \ldots, \alpha_n)} c_{\alpha_1 \ldots \alpha_n} (z_1 - z_1^{(0)})^{\alpha_1} \ldots (z_n - z_n^{(0)})^{\alpha_n}.$$

This new definition of an analytic function is equivalent to the one given on p. 227.

A study of analytic functions of several complex variables reveals properties that have no analogy in the theory of functions of one complex variable. It appears, for instance, that for $n \geqslant 2$ there are domains such that the analyticity of a function in them implies the analyticity of this function in a wider domain. There are no such domains for the case where $n = 1$.

The theory of analytic functions of several complex variables has broad applications in such fields as algebraic geometry and theoretical physics.

7

THE LAURENT SERIES. ISOLATED SINGULAR POINTS OF SINGLE-VALUED ANALYTIC FUNCTIONS. ENTIRE AND MEROMORPHIC FUNCTIONS

7.1 THE LAURENT SERIES

Among the various series that differ from power series the series most closely related to power series in origin and properties are those ordered according to decreasing negative integral powers (exponents) of $z - z_0$, i.e.

$$A_0 + A_1 (z - z_0)^{-1} + A_2 (z - z_0)^{-2} + \ldots + A_n (z - z_0)^{-n} + \ldots . \tag{7.1}$$

If we put $\zeta = 1/(z - z_0)$, we can transform (7.1) into

$$A_0 + A_1\zeta + A_2\zeta^2 + \ldots + A_n\zeta^n + \ldots . \tag{7.2}$$

The radius of convergence of the last series is $R = \dfrac{1}{\overline{\lim\limits_{n\to\infty}} \sqrt[n]{|A_n|}}$.

If $R = 0$, (7.2) converges only at point $\zeta = 0$; if $0 < R < \infty$, it converges absolutely in the circle $|\zeta| < R$ and diverges outside the circle; finally, if $R = \infty$, the series converges absolutely at each point in the finite plane. Returning to the original series (7.1), we can see that it diverges at each finite point if $\overline{\lim\limits_{n\to\infty}} \sqrt[n]{|A_n|} = \infty$, converges absolutely for $|z - z_0| > \overline{\lim\limits_{n\to\infty}} \sqrt[n]{|A_n|}$ and diverges for $|z - z_0| < \overline{\lim\limits_{n\to\infty}} \sqrt[n]{|A_n|}$ if $0 < \overline{\lim\limits_{n\to\infty}} \sqrt[n]{|A_n|} < \infty$, finally converges absolutely at all points of the plane except point $z = z_0$ if $\overline{\lim\limits_{n\to\infty}} \sqrt[n]{|A_n|} = 0$. In other words, the convergence domain of (7.1) is the *exterior* of a circle of radius $r = \overline{\lim\limits_{n\to\infty}} \sqrt[n]{|A_n|}$ centered at z_0 such that for $r = \infty$ it degenerates into the point at infinity, for $0 < r < \infty$ it is the exterior of the circle in the proper sense, and for $r = 0$ it turns into the entire plane without point $z = z_0$. Let us assume that $r = \overline{\lim\limits_{n\to\infty}} \sqrt[n]{|A_n|} < \infty$. Then, indeed, there

exists a convergence domain for (7.1) $|z - z_0| > r$, which we denote by K. Since (7.2) converges uniformly on each closed set of points in the circle k: $|\zeta| < R$ and the linear transformation $\zeta = 1/(z - z_0)$ maps every closed set of points in circle k into a closed set of points in K (this mapping is one-to-one), series (7.1) converges uniformly inside domain K. In this domain it defines a function

$$F(z) = A_0 + A_1 (z - z_0)^{-1} + \ldots + A_n (z - z_0)^{-n} + \ldots \quad (7.1')$$

that is analytic (by Weierstrass's theorem) at all finite points of K. At the point at infinity, $F(z)$ attains the value A_0, i.e. $F(\infty) = A_0$. By definition, we say $F(z)$ is *analytic at the point at infinity*. Thus, the analyticity of a function at the point at infinity depends on whether there exists an expansion of type (7.1′) that converges in a neighborhood of the point at infinity.

The series that generalizes the idea of a power series with only nonnegative integral powers (exponents) of $z - z_0$ or only nonpositive integral powers of $z - z_0$ is the *Laurent series*, called so after the French analyst Paul Laurent. It has the form

$$\sum_{-\infty}^{+\infty} a_n (z - z_0)^n. \quad (7.3)$$

The series is understood to the sum of two series,

$$\sum_{0}^{\infty} a_n (z - z_0)^n \quad \text{and} \quad \sum_{1}^{\infty} a_{-m} (z - z_0)^{-m} \quad (7.4)$$

and is considered if and only if the two converge. Thus, by definition

$$\sum_{-\infty}^{+\infty} a_n (z - z_0)^n = \lim_{\nu \to \infty} \sum_{0}^{\nu} a_n (z - z_0)^n + \lim_{\mu \to \infty} \sum_{1}^{\mu} a_{-m} (z - z_0)^{-m},$$

or, which is the same,

$$\sum_{-\infty}^{+\infty} a_n (z - z_0)^n = \lim_{\substack{\nu \to \infty \\ \mu \to \infty}} \sum_{-\mu}^{\nu} a_n (z - z_0)^n, \quad (7.5)$$

where μ and ν tend to infinity independently. The idea of these notations is as follows: for any positive ε there exists an $N(\varepsilon)$ such that the inequality

$$\left| \sum_{-\infty}^{+\infty} a_n (z - z_0)^n - \sum_{-\mu}^{\nu} a_n (z - z_0)^n \right| < \varepsilon$$

is valid for $\nu > N(\varepsilon)$ and $\mu > N(\varepsilon)$.

The Laurent series is absolutely and uniformly convergent if and only if the two series in (7.4) are such.

We denote $\overline{\lim\limits_{n\to\infty}}\sqrt[n]{|a_n|}$ by λ and $\overline{\lim\limits_{m\to\infty}}\sqrt[n]{|a_{-m}|}$ by r. Then the first series in (7.4) converges absolutely and uniformly inside a domain G that is the interior of a circle Γ: $|z-z_0|=1/\lambda=R$ and diverges in the exterior of the circle, whereas the second series converges absolutely and uniformly inside a domain g that is the exterior of the circle γ: $|z-z_0|=r$ and diverges in the interior of this circle.

The domains G and g have common points if and only if

$$r<R. \tag{7.6}$$

In this case the common part of G and g constitutes an *annulus* D:

$$r<|z-z_0|<R. \tag{7.7}$$

Inside D both series in (7.4) converge absolutely and uniformly and, therefore, so does the Laurent series (7.3), which represents in D an analytic function

$$f(z)=\sum_{\infty}^{+\infty} a_n(z-z_0)^n \quad (r<|z-z_0|<R). \tag{7.8}$$

At each point outside D one of the series in (7.4) converges whereas the other diverges, which implies that outside D the Laurent series diverges. Thus, *the convergence domain of a Laurent series is an annulus** (under condition (7.6)). When speaking of a Laurent series we will always assume that condition (7.6) is met, since without it we cannot define the convergence domain.

If $r<\rho<R$, the series (7.8) is uniformly convergent on the circle γ: $|z-z_0|=\rho$; it will remain uniformly convergent on γ after we multiply all its terms by $(1/2\pi i)(z-z_0)^{-k-1}$, where k is an arbitrary integer. If we integrate the new series along γ, we obtain

$$\frac{1}{2\pi i}\int\limits_{\gamma}\frac{f(z)}{(z-z_0)^{k+1}}\,dz=\sum_{-\infty}^{+\infty} a_n\frac{1}{2\pi i}\int\limits_{\gamma}(z-z_0)^{n-k-1}\,dz.$$

All the integrals on the right-hand side vanish except the one with $n=k$, which is equal to $2\pi i$. (Use the equation for γ in the form $z=z_0+\rho e^{i\theta}$ $(0\leqslant\theta\leqslant 2\pi)$.) Hence,

$$\frac{1}{2\pi i}\int\limits_{\gamma}\frac{f(z)}{(z-z_0)^{k+1}}\,dz=a_k \quad (k=0,\pm 1,\pm 2,\ldots). \tag{7.9}$$

This expresses the expansion coefficients of a Laurent series in terms of the series sum. It follows then that if the sums of two Laurent

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* A Laurent series can also converge at some points on the boundary of the annulus.

series,

$$f(z)=\sum_{-\infty}^{+\infty} a_k (z-z_0)^k \quad \text{and} \quad \varphi(z)=\sum_{-\infty}^{+\infty} b_k (z-z_0)^k,$$

which converge in the annuluses D and Δ containing the same circle $|z - z_0| = \rho$, coincide at the points on this circle, the expansion coefficients of the two series are pairwise equal,

$$a_k = b_k \quad (k = 0, \pm 1, \pm 2, \ldots),$$

i.e. the two series are equivalent. For one, the series are equivalent if D and Δ coincide and $f(z) = \varphi(z)$ at all points of D. This implies that *an expansion into a Laurent series is unique.*

From the uniqueness property of a Laurent expansion we find, in a manner quite similar to that of Sec. 4.3, that if the expansion $\sum_{-\infty}^{+\infty} a_k z^k$ is an even function, all coefficients of odd powers of z are zeros, while if it is odd, the coefficients of even powers of z are zeros.

7.2

THE LAURENT THEOREM

Let us prove the following proposition.

The Laurent theorem (1843). *A function $f(z)$ that is single-valued and analytic in an annulus D: $r < |z - z_0| < R$ is represented in D by a convergent Laurent series:*

$$f(z)=\sum_{-\infty}^{+\infty} a_n (z-z_0)^n.$$

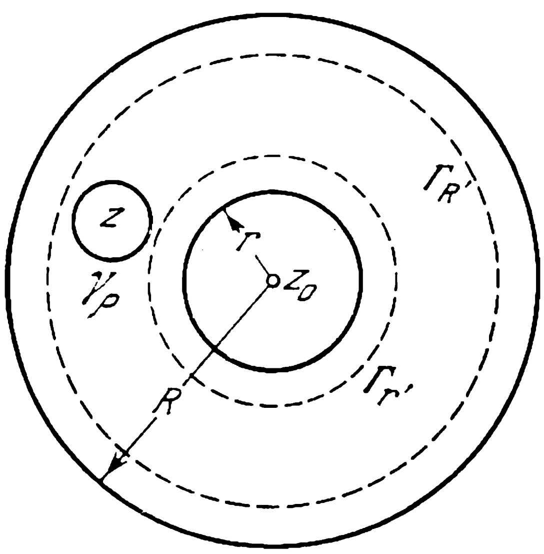

Fig. 49

Note that, under the hypothesis, the annulus may degenerate into a circle with its center deleted ($r = 0$, $R < \infty$) or into the exterior of a circle with the point at infinity deleted ($0 < r$, $R = \infty$) or, finally, into the entire plane with points z_0 and ∞ deleted ($r = 0$, $R = \infty$). These cases are not excluded from the following proof.

Let z be a point of D. We build a new annulus D': $r' < |\zeta - z_0| < R'$ that lies inside the original annulus and contains point z (Fig. 49). To this end it suffices to take

$$r < r' < |z - z_0| < R' < R.$$

Suppose, in addition, that $|\zeta - z| = \rho$, is a circle centered at z and lying inside D'. Since

$$\varphi(\zeta) = \frac{f(\zeta)}{\zeta - z}$$

is an analytic function of ζ in D except at point $\zeta = z$, Cauchy's integral theorem for a composite contour (see Sec. 5.9) states that

$$\frac{1}{2\pi i} \int\limits_{\Gamma_{R'}} \frac{f(\zeta)}{\zeta - z}\, d\zeta = \frac{1}{2\pi i} \int\limits_{\Gamma_{r'}} \frac{f(\zeta)}{\zeta - z}\, d\zeta + \frac{1}{2\pi i} \int\limits_{\gamma_\rho} \frac{f(\zeta)\, d\zeta}{\zeta - z}, \qquad (7.10)$$

where $\Gamma_{R'}$, $\Gamma_{r'}$, and γ_ρ are, respectively, the circles

$$|\zeta - z_0| = R', \quad |\zeta - z_0| = r', \quad \text{and} \quad |\zeta - z_0| = \rho,$$

which are traversed counterclockwise in the integrations. But the last integral on the right-hand side of (7.10) is Cauchy's integral and, hence, is simply $f(z)$. Therefore,

$$f(z) = \frac{1}{2\pi i} \int\limits_{\Gamma_{R'}} \frac{f(\zeta)}{\zeta - z}\, d\zeta - \frac{1}{2\pi i} \int\limits_{\Gamma_{r'}} \frac{f(\zeta)\, d\zeta}{\zeta - z}. \qquad (7.11)$$

Let us write the fraction $1/(\zeta - z)$ in the first integral ($\zeta \in \Gamma_{R'}$) in the form of the sum of a geometric series with a ratio $(z - z_0)/(\zeta - z_0)$ whose modulus is

$$\left| \frac{z - z_0}{\zeta - z_0} \right| = \frac{|z - z_0|}{R'} = \theta < 1.$$

This yields

$$\frac{1}{\zeta - z} = \frac{1}{\zeta - z_0 - (z - z_0)} = \frac{1}{\zeta - z_0} \frac{1}{1 - \dfrac{z - z_0}{\zeta - z_0}} = \sum_0^\infty \frac{(z - z_0)^n}{(\zeta - z_0)^{n+1}}. \qquad (7.12)$$

Since for any point $\zeta \in \Gamma_{R'}$ we have

$$\left| \frac{(z - z_0)^n}{(\zeta - z_0)^{n+1}} \right| = \frac{1}{R'} \theta^n \quad (0 < \theta < 1),$$

the series (7.12) converges uniformly on $\Gamma_{R'}$ (with respect to ζ) and so does the series obtained by multiplying each term in (7.12) into $(1/2\pi i) f(\zeta)$ (this function's modulus is bounded in $\Gamma_{R'}$):

$$\frac{1}{2\pi i} \frac{f(\zeta)}{\zeta - z} = \sum_0^\infty \frac{1}{2\pi i} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} (\zeta - z_0)^n.$$

This implies that we can integrate the last series termwise in $\Gamma_{R'}$, which yields

$$\frac{1}{2\pi i} \int\limits_{\Gamma_{R'}} \frac{f(\zeta)}{\zeta - z}\, d\zeta = \sum_0^\infty a_n (z - z_0)^n, \qquad (7.13)$$

where

$$a_n = \frac{1}{2\pi i} \int\limits_{\Gamma_{R'}} \frac{f(\zeta)\, d\zeta}{(\zeta - z_0)^{n+1}} \quad (n = 0, 1, 2, \ldots). \tag{7.14}$$

We have, therefore, expanded the first term on the right-hand side of (7.11) in a power series with nonnegative powers of $z - z_0$.

We now turn to the second term in (7.11). We write the fraction $-1/(\zeta - z)$ ($\zeta \in \Gamma_{r'}$) in the form of the sum of a geometric series with a ratio $(\zeta - z_0)/(z - z_0)$ whose modulus is

$$\left| \frac{\zeta - z_0}{z - z_0} \right| = \frac{r'}{| z - z_0 |} = \vartheta < 1.$$

This yields

$$-\frac{1}{\zeta - z} = \frac{1}{z - z_0} \frac{1}{1 - \frac{\zeta - z_0}{z - z_0}} = \sum_0^{\infty} \frac{(\zeta - z_0)^n}{(z - z_0)^{n+1}}. \tag{7.15}$$

Since this series is uniformly convergent on $\Gamma_{r'}$, we can multiply each term into $(1/2\pi i)\, f(\zeta)$ and then integrate the new series termwise. This yields

$$-\frac{1}{2\pi i} \int\limits_{\Gamma_{r'}} \frac{f(\zeta)}{\zeta - z}\, d\zeta = \sum_1^{\infty} a_{-n} (z - z_0)^{-n}, \tag{7.16}$$

where

$$a_{-n} = \frac{1}{2\pi i} \int\limits_{\Gamma_{r'}} \frac{f(\zeta)\, d\zeta}{(\zeta - z_0)^{-n+1}} \quad (n = 1, 2, \ldots). \tag{7.17}$$

We have, therefore, expanded the second term on the right-hand side of (7.11) in a power series with negative powers of $z - z_0$.

Substituting the two expansions, (7.13) and (7.16), into the right-hand side of (7.11), we arrive at an expansion of $f(z)$ in a Laurent series at an arbitrary point $z \in D$:

$$f(z) = \sum_0^{\infty} a_n (z - z_0)^n + \sum_1^{\infty} a_{-n} (z - z_0)^{-n} = \sum_{-\infty}^{+\infty} a_n (z - z_0)^n. \tag{7.18}$$

The expansion coefficients can be calculated partly via (7.14) and partly via (7.17). If we take an arbitrary circle Γ: $| z - z_0 | = \lambda$, where $r < \lambda < R$, Cauchy's integral theorem for a composite contour shows that each coefficient can be calculated by integration along the circle Γ:

$$a_n = \frac{1}{2\pi i} \int\limits_{\Gamma} \frac{f(\zeta)\, d\zeta}{(\zeta - z_0)^{n+1}} \quad (n = 0, \pm 1, \pm 2, \ldots). \tag{7.19}$$

This result agrees with that obtained earlier (see (7.9)).

7.3
ISOLATED SINGULAR POINTS OF SINGLE-VALUED ANALYTIC FUNCTIONS

Consider a single-valued function $f(z)$ that is analytic in the neighborhood of a point z_0 with the exception, perhaps, of this point. Then $f(z)$ is analytic in a domain D:

$$0 < |z - z_0| < R.$$

With respect to point z_0 we can make two assumptions. First, there may exist a finite complex number a_0 such that, if we put $f(z_0) = a_0$, we have a function $f(z)$ analytic in the entire circle $|z - z_0| < R$ (point z_0 inclusive). Second, there may be no such number. In the first case we speak of z_0 as a *regular point* of $f(z)$ and the function $f(z)$ as *regular* at this point. In the second, z_0 is said to be an *isolated singular point* or simply an *isolated singularity*.

The main method of studying analytic functions in the neighborhood of an isolated singularity z_0 is to expand the function in a Laurent series. Let us apply the Laurent theorem to a function $f(z)$ in the domain D: $0 < |z - z_0| < R$. This domain is a degenerated annulus with a zero inner radius. We have

$$f(z) = \sum_{-\infty}^{+\infty} a_n (z - z_0)^n \quad (z \in D), \tag{7.20}$$

where

$$a_n = \frac{1}{2\pi i} \int_{\gamma_\rho} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta \quad (n = 0, \pm 1, \pm 2, \ldots)$$

and γ_ρ is a circle centered at z_0 and with a radius ρ $(0 < \rho < R)$. We can also use expansion (7.20) when z_0 is a regular point. In this case the Laurent series turns into a Taylor series and we have $a_{-1} = a_{-2} = \ldots = 0$.

Let us prove an important

Theorem. *For a function $f(z)$ that is single-valued and analytic in a domain D: $0 < |z - z_0| < R$ to be regular at point z_0 it is necessary and sufficient that there be a neighborhood U of z_0 in which the modulus of $f(z)$ is bounded.*

Proof. Suppose that point z_0 is regular for $f(z)$. Then we can define $f(z)$ at z_0 in a way such that it is analytic and, hence, bounded in a neighborhood of z_0. Hence the necessity of the hypothesis.

Now we prove its sufficiency. Suppose there exists a neighborhood U of point z_0 and a positive number $M < \infty$ such that

$$|f(z)| < M \quad \text{for all} \quad z \in U.$$

Then, if we choose a ρ $(0 < \rho < R)$ such that the circle γ_ρ belongs to U, we arrive via (7.19) at the following relations for the moduli

of the expansion coefficients in the Laurent series (7.20):

$$|a_n| < \frac{1}{2\pi} M \frac{2\pi\rho}{\rho^{n+1}}, \text{ i.e. } |a_n| < \frac{M}{\rho^n}.$$

Let us study only the coefficients of negative powers of $z - z_0$, i.e. with negative n's. If we tend ρ to zero (bearing in mind that $0 < \rho < R$), we obviously obtain

$$a_n = 0 \quad \text{for} \quad n = -1, -2, -3, \ldots.$$

Hence, the Laurent series turns into the Taylor series

$$f(z) = \sum_0^{\infty} a_n (z - z_0)^n.$$

The last step is to put $f(z_0) = a_0$, which makes point z_0 regular for $f(z)$. This completes the proof.

From the above theorem it follows that for a point z_0 to be an isolated singularity for $f(z)$ it is necessary and sufficient that in any neighborhood of this point the modulus of $f(z)$ be infinite, i.e.

$$\overline{\lim_{z \to z_0}} |f(z)| = +\infty. \tag{7.21}$$

This implies a priori that there are two possibilities for the behavior of a function $f(z)$ in the neighborhood of an isolated singularity:

(a) $\lim\limits_{z \to z_0} f(z) = \infty$;

(b) there is neither a finite nor an infinite limit for $f(z)$ as z tends to z_0.

Each of these cases does indeed realize itself.

Suppose $f(z) = 1/(z - z_0)^n$ with n a positive integer. Obviously, this function is analytic when $0 < |z - z_0|$ and $\lim\limits_{z \to z_0} f(z) = \infty$. Here we are dealing with case (a).

Another example is the function $f(z) = e^{1/(z - z_0)}$. It is also analytic when $0 < |z - z_0|$ but, in contrast to the previous example, no limit exists as $z \to z_0$. Indeed, if, for instance, point z lies on a straight line that passes through z_0 and is parallel to the real axis so that $z - z_0 = x - x_0$ is a real number, then for $x > x_0$ the function $e^{1/(x - x_0)} \to \infty$ as $x \to x_0$, and for $x < x_0$ the function $e^{1/(x - x_0)} \to 0$ as $x \to x_0$. Here we are dealing with case (b). Figure 50 depicts the surface $u = |\exp(1/z)|$, the modular surface of the discussed function (with $z_0 = 0$).

An isolated singular point z_0 at which condition (a) is met, i.e. $\lim\limits_{z \to z_0} f(z) = \infty$, is called a *pole* of the analytic function.

An isolated singular point z_0 at which condition (b) is met, i.e. there is no limit for $f(z)$ as $z \to z_0$, is called an *essential singularity*.

Let us study each of these two cases in detail.

Suppose z_0 is a pole of $f(z)$. Then $\lim\limits_{z \to z_0} |f(z)| = \infty$ and, hence, there is a neighborhood $|z - z_0| < \delta < R$ of point z_0 in which $|f(z)| > 1$. In this neighborhood the function $\varphi(z) = 1/f(z)$ is, obviously, analytic everywhere except, perhaps, at point z_0. But from the fact that $|\varphi(z)| = 1/|f(z)| < 1$ it follows, by the theorem

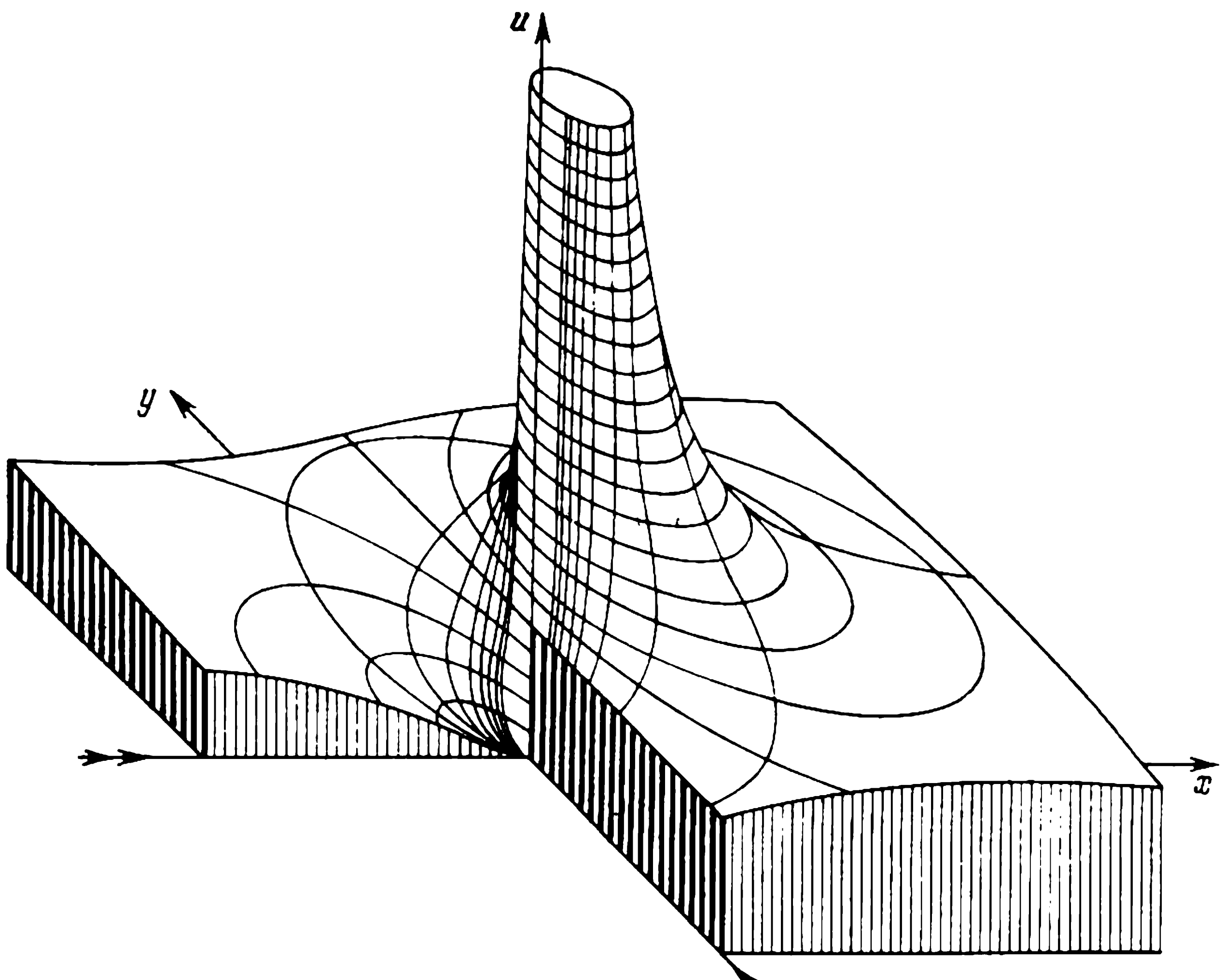

Fig. 50

we proved earlier in this section, that z_0 is a regular point for $\varphi(z)$. The value of this function at z_0 is $\lim\limits_{z \to z_0} \frac{1}{f(z)} = 0$. Whence point z_0 is a zero of $\varphi(z)$.

Conversely, if we know that $\varphi(z)$ is a function single-valued and analytic in a neighborhood of point z_0 and that this point is a zero of the function, with $\varphi(z) \not\equiv 0$, then there exists a positive Δ so small that $\varphi(z)$ has no other zeros except z_0 in the circle $|z - z_0| < \Delta$ (see Sec. 6.11). Let us build the function $f(z) = 1/\varphi(z)$; it is single-valued and analytic for $0 < |z - z_0| < \Delta$ and tends to ∞ as $z \to z_0$. Hence, z_0 is a pole of $f(z)$.

Therefore, we have proved the following proposition: *a point z_0 is a pole of a function $f(z)$ if and only if it is a zero of the function $\varphi(z) = 1/f(z)$.*

In view of this property, which establishes a relationship between zeros and poles, we can introduce the notion of the order of a pole. We will say that the *order* of a pole z_0 of a function $f(z)$ is k ($k \geqslant 1$) if point z_0 is a zero of the kth order of the function $1/f(z)$. If $k = 1$, the pole is said to be *simple*.

In the neighborhood of a pole of an order k the Laurent expansion behaves in a certain way; namely, there is the following

Theorem. *A point z_0 is a pole of the kth order of a function $f(z)$ if and only if the Laurent expansion for $f(z)$ in a neighborhood of z_0 does not contain terms with powers lower than $-k$ and the expansion coefficient of $(z - z_0)^{-k}$ is nonzero.* In other words, the Laurent expansion must be

$$f(z) = a_{-k}(z - z_0)^{-k} + \ldots + a_{-1}(z - z_0) + a_0 + a_1(z - z_0) + \ldots, \tag{7.22}$$

where $a_{-k} \neq 0$.

Indeed, suppose that z_0 is a kth order pole of $f(z)$. Then $1/f(z)$ must have at this point a zero of the kth order, whence

$$\frac{1}{f(z)} = A_k(z - z_0)^k + A_{k+1}(z - z_0)^{k+1} + \ldots \quad (A_k \neq 0)$$

in a neighborhood of z_0. Therefore,

$$f(z) = \frac{1}{(z - z_0)^k} \frac{1}{A_k + A_{k+1}(z - z_0) + \ldots}. \tag{7.23}$$

The power series $A_k + A_{k+1}(z - z_0) + \ldots$ represents an analytic function that does not vanish in a neighborhood of z_0 (since $A_k \neq 0$). Hence, the function $\dfrac{1}{A_k + A_{k+1}(z - z_0) + \ldots}$ is analytic in a neighborhood of z_0 and admits an expansion of the type $\alpha_0 + \alpha_1(z - z_0) + \ldots$, where $\alpha_0 = 1/A_k \neq 0$. Substituting this last series into (7.23), we arrive at the following expansion for $f(z)$:

$$f(z) = \alpha_0(z - z_0)^{-k} + \alpha_1(z - z_0)^{-k+1} + \ldots \quad (\alpha_0 \neq 0),$$

which in view of the uniqueness of a Laurent expansion is exactly that for $f(z)$. To within notations it coincides with (7.22) ($\alpha_n = a_{n-k}$, $n = 0, 1, 2, \ldots$). Hence the necessity of the hypothesis.

We will now prove that it is sufficient. Suppose $f(z)$ in a neighborhood of z_0 can be expanded into (7.22) with $a_{-k} \neq 0$. If we write this expansion in the form

$$f(z) = \frac{a_{-k} + a_{-k+1}(z - z_0) + \ldots}{(z - z_0)^k},$$

we can conclude that

$$\frac{1}{f(z)} = (z - z_0)^k \frac{1}{a_{-k} + a_{-k+1}(z - z_0) + \dots},$$

or, by substituting for $\dfrac{1}{a_{-k} + a_{-k+1}(z - z_0) + \dots}$ its expansion in a Taylor series in powers of $z - z_0$,

$$\frac{1}{f(z)} = (z - z_0)^k [\beta_0 + \beta_1 (z - z_0) + \dots]$$
$$= \beta_0 (z - z_0)^k + \beta_1 (z - z_0)^{k+1} + \dots,$$

where $\beta_0 = 1/a_{-k} \neq 0$.

We see that point z_0 is a zero of the kth order of the function $1/f(z)$. Therefore, the same point is a pole of the kth order of $f(z)$. The proof is complete.

7.4 THE SOCHOZKI-CASORATI-WEIERSTRASS THEOREM

We now turn to the case of an essential singularity. The behavior of a function in the neighborhood of an essential singularity is determined by

The Sochozki-Casorati-Weierstrass theorem (1868-76). *For any complex number A (proper or improper) there exists a sequence of points $\{z_n\}$ converging to an essential singularity z_0 such that* $\lim\limits_{n\to\infty} f(z_n) = A$.

Proof. When $A = \infty$, the theorem is valid, since the modulus of $f(z)$ is not bounded in any neighborhood of an essential singularity. Suppose now that $A \neq \infty$; we prove the theorem by contradiction. If in an arbitrary neighborhood of point z_0 there are no points at which the values of $f(z)$ are arbitrarily close to A, then there must exist a neighborhood $0 < |z - z_0| < \delta$ and a positive α such that $|f(z) - A| > \alpha$ for $0 < |z - z_0| < \delta$. We build the function $\varphi(z) = 1/(f(z) - A)$, which is analytic in the neighborhood $0 < < |z - z_0| < \delta$. Besides, in this neighborhood,

$$|\varphi(z)| = \frac{1}{|f(z) - A|} < \frac{1}{\alpha}.$$

Therefore, by the first theorem of Sec. 7.3 $\varphi(z)$ is regular at point z_0 and its value at this point must be equal to $\lim\limits_{z\to z_0} \dfrac{1}{f(z) - A}$. But $f(z)$ is not bounded in any neighborhood of point z_0. Whence, the above limit may only be zero, i.e. $\varphi(z_0) = 0$. Hence, the function $1/(f(z) - A)$ has a zero at z_0, which implies that $f(z) - A$ and, hence, $f(z)$ have a pole at this point. We have arrived at a contradiction with the hypothesis. This completes the proof.

Here are two examples to illustrate the theorem.

(a) $f(z) = \sin(1/z)$. The origin of coordinates is the essential singularity. Indeed, as z tends to zero, $\sin(1/z)$ tends neither to a finite nor to an infinite limit, which is immediately apparent if we take only real values of z.

If $A = \infty$, then, by assuming, for instance, that $z_n = i/n$ and, hence, $1/z_n = -in$, we obtain

$$\sin\frac{1}{z_n} = -i\sinh n \to \infty \text{ as } n \to \infty.$$

Now suppose that $A \neq \infty$. To build the sequence $\{z_n\}$ required by the Sochozki-Casorati-Weierstrass theorem, we solve the equation

$$\sin\frac{1}{z} = A.$$

This yields

$$\frac{1}{z} = \operatorname{Arc}\sin A = \frac{1}{i}\operatorname{Ln}(iA + \sqrt{1-A^2}),$$

whence

$$z = \frac{i}{\operatorname{Ln}(iA+\sqrt{1-A^2})} = \frac{i}{\ln(iA+\sqrt{1-A^2})+2k\pi i}.$$

If we assume that

$$z_n = \frac{i}{\ln(iA+\sqrt{1-A^2})+2n\pi i}$$

and set n equal to 1, 2, 3, . . ., we arrive at a sequence $\{z_n\}$ that converges to zero and meets the condition

$$f(z_n) = A \quad (n = 1, 2, \ldots).$$

Hence, $\lim\limits_{n\to\infty} f(z_n) = A.$

(b) $f(z) = e^{1/z}$. Here the essential singularity is again the origin of coordinates, so that once more there does not exist a limit for $e^{1/z}$ as $z \to 0$.

If we assume that $A = \infty$, we take $z_n = 1/n$. This yields $f(z_n) = e^n \to \infty$ as $n \to \infty$, i.e. the sequence $\{1/n\}$ does indeed meet the requirement of the Sochozki-Casorati-Weierstrass theorem when $A = \infty$. Now suppose $A = 0$. Assuming that $z_n = -1/n$, we have $f(z_n) = e^{-n} \to 0$ as $n \to \infty$, i.e. the requirement of the theorem is met in this case as well. Finally, suppose that A is neither ∞ nor zero. The simplest way to select the points z_n is to solve the equation

$$e^{1/z} = A.$$

This yields

$$\frac{1}{z} = \operatorname{Ln} A,$$

whence

$$z = \frac{1}{\operatorname{Ln} A} = \frac{1}{\ln A + 2k\pi i}.$$

If we assume that

$$z_n = \frac{1}{\ln A + 2n\pi i} \quad (n = 1, \ldots),$$

we arrive at a sequence $\{z_n\}$ that converges to zero and meets the condition $f(z_n) = A$. Hence, $\lim\limits_{n\to\infty} f(z_n) = A$.

From the above theorem it follows that if z_0 is an essential singularity of the function $f(z)$ and E_δ is the set of values that the function admits in an arbitrarily small neighborhood $|z - z_0| < \delta$ of this point, then the closure of E_δ (i.e. the set E_δ and all the limit (accumulation) points of E_δ) coincides with the extended complex plane.

Indeed, each point A in the complex plane is the limit for a sequence $\{f(z_n)\}$ of points belonging to E_δ; hence, A belongs to the closure of E_δ.

In the above examples (a) and (b) we have seen that, with some exceptions ($A = \infty$ in the first example and $A = \infty$ and $A = 0$ in the second), instead of sequence of points $\{z_n\}$ for which there is the *limiting relation*

$$\lim_{z_n\to z_0} f(z_n) = A$$

we can find such a sequence for which the *exact relation*

$$f(z_n) = A \quad (n = 1, 2, \ldots)$$

is valid. It turns out that we can make a similar statement in the general case as well. To this end we have

Picard's second theorem. *If z_0 is an essential singularity of a function $f(z)$, then for each $A \neq \infty$ with the exception of, perhaps, one value $A = A_0$ there exists an infinite sequence of A-points of $f(z)$ that converges to z_0.**

In the example with $f(z) = \sin(1/z)$ there is no such exceptional value, and in the example with $f(z) = e^{1/z}$ this value is zero since the function always differs from zero. It is easy to prove that the Sochozki-Casorati-Weierstrass theorem is a corollary of Picard's second theorem.

The last theorem implies that the set of values that $f(z)$ admits in an arbitrary neighborhood $|z - z_0| < \delta$ of an essential singularity z_0 coincides with the entire finite complex plane $|z| < \infty$ except, at the most, at one point A_0 (A_0 does not depend on δ).

The Laurent expansion of a function $f(z)$ in a neighborhood of an essential singularity z_0 must have an infinite number of terms with

* See, for example, A. I. Markushevich, *The Theory of Analytic Functions* [in Russian], vol. 2, Nauka, Moscow, 1968, Chap. 8, Sec. 8.4.

negative powers of $z - z_0$ (i.e. the expansion coefficients of these terms are all nonzero). Indeed, if there were no such terms in the Laurent expansion in this case, z_0 would be a regular point of $f(z)$, and if there were only a finite number of such terms, z_0 would be a pole of $f(z)$ (by the second theorem of Sec. 7.3). Conversely, *when a Laurent expansion of a function* $f(z)$ *in a neighborhood of a point* z_0 *contains an infinite number of terms with negative powers of* $z - z_0$, *the point* z_0 *is an essential singularity.* Indeed, this point cannot be a regular point of $f(z)$ (since in this case there would be no terms with negative powers) or a pole (there would be a finite number of terms with negative powers).

As an example we take the function $\exp(1/z^4)$. For any $z \neq 0$ the following expansion is true:

$$\exp\frac{1}{z^4} = 1 + \frac{1}{z^4} + \frac{1}{2!z^8} + \frac{1}{3!z^{12}} + \dots .$$

Obviously, this can be considered to be the Laurent expansion of this function in a neighborhood of point $z = 0$. Since it contains an infinite number of terms with negative powers of z, the point $z = 0$ is an essential singularity. We arrive at the same result if we investigate the behavior of $\exp(1/z^4)$ in the neighborhood of the origin of coordinates. The reader can easily see that $\exp(1/z^4) \to \infty$ as $z \to 0$ along the coordinate axes and tends to zero as $z \to 0$ along the bisectors of the angles between the axes of coordinates. Therefore, $\exp(1/z^4)$ has no limit as $z \to 0$, and point $z = 0$ is an essential singularity of $\exp(1/z^4)$.

The above discussion implies that the nature of an isolated singularity is determined chiefly by the terms with negative powers in the Laurent expansion of the particular function in a (small) neighborhood of this point. For this reason the series $\sum_{1}^{\infty} a_{-k}(z - z_0)^{-k}$ is called the *principal part* of the Laurent expansion $\sum_{-\infty}^{+\infty} a_k (z - z_0)^k$ at z_0, or the *singular part* of $f(z)$ at a. The series $\sum_{0}^{\infty} a_k (z - z_0)^k$, which consists of terms with nonnegative powers, represents a function that is regular at z_0 and is therefore called the *regular*, or *holomorphic*, *part* of $f(z)$, or of the Laurent expansion.

In applying these propositions we must bear in mind that we are dealing with Laurent expansions that converge in a neighborhood $0 < |z - z_0| < R$ of the point under study.

Let us take the Laurent series

$$\dots + \frac{1}{z^n} + \frac{1}{z^{n-1}} + \dots + \frac{1}{z} + \frac{1}{2} + \frac{z}{2^2} + \dots + \frac{z^n}{2^{n+1}} + \dots$$

as an example. It contains an infinite number of terms with negative powers of z. But before we state that $z = 0$ is an essential singularity of the series sum, we must examine whether the series converges in a neighborhood of the point. Note that the above series is a sum of two progressions, $\sum_{1}^{\infty} z^{-n}$ and $\sum_{0}^{\infty} \frac{z^n}{2^{n+1}}$. The first converges for $|z| > 1$ and represents the function $\frac{\frac{1}{z}}{1-\frac{1}{z}} = \frac{1}{z-1}$; the second converges for $|z| < 2$ and represents the function $\frac{\frac{1}{2}}{1-\frac{z}{2}} = \frac{1}{2-z}$. Hence, the convergence domain of the series at hand is annulus $1 < |z| < 2$, which is, obviously, not a neighborhood of the origin of coordinates. The sum of the series in this annulus is $\frac{1}{z-1} + \frac{1}{2-z} = \frac{-1}{z^2-3z+2}$, a function that is regular at the origin of coordinates and whose isolated singularities are two simple poles, $z = 1$ and $z = 2$.

7.5

SINGULAR POINTS OF DERIVATIVES AND RATIONAL COMBINATIONS OF ANALYTIC FUNCTIONS

To speed up the definition of position and type of singular points of concrete functions it is wise to bear in mind the simple propositions that follow from the theorems of Secs. 7.3 and 7.4.

(a) If $f(z)$ and $\varphi(z) \not\equiv 0$ are functions single-valued and analytic in a given domain G, the function $F(z) = f(z)/\varphi(z)$ may have in G singular points, namely poles, only at the zero points of $\varphi(z)$. Suppose that ζ is a zero point of the kth order of $\varphi(z)$ $(k \geqslant 1)$ and a zero point of the lth order of $f(z)$ $(l \geqslant 0)$ (in the case where ζ is not a zero point of $f(z)$ we assume that $l = 0$). Then in a neighborhood of ζ we have

$$F(z) = \frac{\frac{f^{(l)}(\zeta)}{l!}(z-\zeta)^l + \dots}{\frac{\varphi^{(k)}(\zeta)}{k!}(z-\zeta)^k + \dots} = (z-\zeta)^{l-k}\,\frac{\frac{f^{(l)}(\zeta)}{l!} + \dots}{\frac{\varphi^{(k)}(\zeta)}{k!} + \dots},$$

where $f^{(l)}(\zeta) \neq 0$ and $\varphi^{(k)}(\zeta) \neq 0$. This implies that at point ζ the function $F(z)$ has a pole of the $(k-l)$th order if $k > l$ and is regular if $k \leqslant l$; the point is a zero point of $F(z)$ of the $(l-k)$th order if $k < l$.

(b) If $f(z)$ and $\varphi(z)$ are two functions that have no other singular points in a domain G except poles, their sum, difference, product, and quotient (the last has meaning only if $\varphi(z) \not\equiv 0$) have no other singular points except poles.

For instance, consider the difference $f(z) - \varphi(z)$ and suppose ζ is a point in whose neighborhood the Laurent expansions of $f(z)$ and $\varphi(z)$ are

$$f(z) = \frac{a_{-l}}{(z-\zeta)^l} + \ldots + \frac{a_{-1}}{z-\zeta} + a_0 + a_1(z-\zeta) + \ldots,$$

$$\varphi(z) = \frac{b_{-k}}{(z-\zeta)^k} + \ldots + \frac{b_{-1}}{z-\zeta} + b_0 + b_1(z-\zeta) + \ldots .$$

Here by l and k we denote the order of point ζ, which is a pole of $f(z)$ or $\varphi(z)$. For greater generality we assume that $l \leqslant 0$ (or $k \leqslant 0$) when point ζ is a regular point of $f(z)$ (or $\varphi(z)$). In this case we start the expansion with terms with nonnegative powers of $z - \zeta$.

If we subtract termwise the expansion for $\varphi(z)$ from that for $f(z)$, we arrive at the expansion for $f(z) - \varphi(z)$. Obviously, point ζ is a pole of this difference if and only if it is a pole of at least one of the functions $f(z)$ and $\varphi(z)$ ($k \geqslant 1$ or $l \geqslant 1$), and the principal parts of the expansions for $f(z)$ and for $\varphi(z)$ do not coincide. But if they do (i.e. $k = l$, $a_{-k} = b_{-k}, \ldots, a_{-1} = b_{-1}$), we have the following expansion for the difference:

$$f(z) - \varphi(z) = a_0 - b_0 + (a_1 - b_1)(z - \zeta) + \ldots,$$

which implies that ζ is a regular point of $f(z) - \varphi(z)$.

Next we consider the function

$$F(z) = \frac{f(z)}{\varphi(z)} \quad (\varphi(z) \not\equiv 0).$$

Its singular points can coincide only with the zeros of $\varphi(z)$ or the poles of $f(z)$. Suppose that ζ is a point that is a zero (a pole) of $f(z)$ (of $\varphi(z)$). In both cases we can write

$$f(z) = (z - \zeta)^l [a_0 + a_1(z - \zeta) + \ldots],$$
$$\varphi(z) = (z - \zeta)^k [b_0 + b_1(z - \zeta) + \ldots],$$

where l and k are integers (positive, negative, or zeros), and the expressions in the brackets are power series that converge in a neighborhood of point ζ and have nonzero free terms ($a_0 \neq 0$ and $b_0 \neq 0$). If $k > 0$, for instance, then $\varphi(z)$ has a kth order zero at point ζ; if $k = 0$, then point $z = \zeta$ is a regular point at which $\varphi(z) \neq 0$; and, finally, if $k < 0$, then $z = \zeta$ is a singular point of $\varphi(z)$, namely, a pole of the $-k$th order.

If we substitute the expansions for $f(z)$ and $\varphi(z)$ into the expression for $F(z)$, we obtain

$$F(z) = (z-\zeta)^{l-k} \frac{a_0 + a_1 (z-\zeta) + \ldots}{b_0 + b_1 (z-\zeta) + \ldots}.$$

Obviously, for $l \geqslant k$ point ζ is a regular point of $F(z)$ (for one, for $l > k$ it is an $(l-k)$th order zero) and for $l < k$ it is a pole of the $(k-l)$th order of $F(z)$.

(c) Suppose that $f(z)$ is a single-valued function with no other singular points in a domain G except poles. Then the derivative $f'(z)$ cannot have in G any singularities except poles. Namely, $f'(z)$ has a pole at each point where $f(z)$ has one, and the order of the pole is higher than that of the pole of the original function by unity.

Indeed, suppose ζ is a pole of $f(z)$ of order $k \geqslant 1$. Then we can expand $f(z)$ in a neighborhood $U: 0 < |z-\zeta| < R$ of point ζ in a Laurent series:

$$f(z) = \frac{A_{-k}}{(z-\zeta)^k} + \ldots + \frac{A_{-1}}{z-\zeta} + A_0 + A_1 (z-\zeta) + \ldots \quad (A_{-k} \neq 0).$$

Since the terms of this expansion are analytic inside U and the expansion converges uniformly inside U (which is a property of a Laurent series), we can differentiate the expansion termwise. This yields

$$f'(z) = -\frac{kA_{-k}}{(z-\zeta)^{k+1}} - \ldots - \frac{A_{-1}}{(z-\zeta)^2} + A_1 + \ldots \quad (kA_{-k} \neq 0).$$

This is the Laurent expansion of $f'(z)$ in the neighborhood of point ζ, which implies that ζ is a pole of the $(k+1)$st order of $f'(z)$.

(d) Suppose that $f(z) \not\equiv \text{const}$ is a single-valued function without any singularities in a domain G except poles and suppose $A \neq \infty$ is an arbitrary complex number. Then the *logarithmic derivative* of the function $f(z) - A$,

$$\frac{d\{\mathrm{Ln}\,[f(z)-A]\}}{dz} = \frac{f'(z)}{f(z)-A},$$

has no singularities in G except poles. Namely, the logarithmic derivative has simple poles at points where $f(z)$ has poles and at all A-points of $f(z)$ (i.e. at all zeros of $f(z) - A$).

To verify this statement let us refer to the general case studied in item (b). There we saw that the quotient of two functions can have poles at the zero points of the denominator or at the poles of the numerator. Suppose that $z = \zeta$ is a kth order zero of the denominator, i.e. an A-point of $f(z)$ of the kth order. Then

$$f(z) - A = a_0 (z-\zeta)^k + a_1 (z-\zeta)^{k+1} + \ldots \quad (k \geqslant 1,\ a_0 \neq 0),$$

which implies that

$$f'(z) = ka_0(z-\zeta)^{k-1} + a_1(k+1)(z-\zeta)^k + \ldots,$$

i.e. point ζ is a $(k-1)$st order zero of the numerator of the quotient. From this it follows that ζ is a simple pole of the logarithmic derivative.

On the other hand, suppose $z = \zeta$ is a pole of the numerator $f'(z)$. This is possible only if ζ is a pole of $f(z) - A$. Then, as we saw in item (c), the order of the pole of $f'(z)$ is a unit higher than that of the same pole of $f(z) - A$. Hence, the logarithmic derivative does have a simple pole at ζ.

(e) If ζ is a regular point or pole of a function $f(z) \not\equiv 0$ and an essential singularity of a function $\varphi(z)$, this point is an essential singularity of any of the combinations $\varphi(z) \pm f(z)$, $f(z)\varphi(z)$, and $\varphi(z)/f(z)$.

Indeed, if we denote these combinations by $\psi_1(z)$, $\psi_2(z)$, and $\psi_3(z)$, respectively, then

$$\varphi(z) = \psi_1(z) \pm f(z), \quad \varphi(z) = \frac{\psi_2(z)}{f(z)}, \text{ and } \varphi(z) = \psi_3(z)f(z).$$

If we assume that $\psi_j(z)$ $(j = 1, 2, 3)$ is regular or has a pole at $z = \zeta$, the function $\varphi(z)$ will also be regular or have a pole at the same point, which contradicts the initial assumption. Point $z = \zeta$ cannot, therefore, be a regular point or a pole of $\psi_j(z)$. Since these functions are single-valued and analytic in a neighborhood of ζ except at this point, the point must be an isolated singularity of $\psi_j(z)$. But since it cannot be a pole, it is an essential singularity of the $\psi_j(z)$.

(f) If ζ is an essential singularity of a function $\varphi(z)$, then either $1/\varphi(z)$ can have an essential singularity at ζ or the point may be a *nonisolated singularity*, a *limit point of a sequence of poles*.

Indeed, there are two possibilities here. Either there exists a neighborhood of ζ where $\varphi(z)$ does not vanish or there is no such neighborhood. In the first case, the function $\psi(z) = 1/\varphi(z)$ is analytic in a neighborhood of point ζ except at the point itself. This point can be neither a regular point nor a pole of $\psi(z)$; otherwise, it would be a pole or a regular point of $\varphi(z) = 1/\psi(z)$, which contradicts the initial assumption. Hence, ζ is an essential singularity of $\psi(z)$.

In the second case, in every neighborhood of ζ there are zeros of $\varphi(z)$ and, hence, in every neighborhood there are poles of $\psi(z) = 1/\varphi(z)$. This implies that every neighborhood of point ζ has singularities (namely, poles) of $\psi(z)$ and, therefore, ζ in this case is a nonisolated singularity of $\psi(z)$, a limit point of a sequence of poles.

7.6
SINGULARITIES AT INFINITY

Let us consider a single-valued function $f(z)$ that is analytic in the exterior $|z| > r$ of a circle centered at the origin of coordinates with the exception, perhaps, of the point at infinity. If we introduce the transformation $z = 1/\zeta$, we can reduce the study of such a function to that of a function $f^*(\zeta) = f(1/\zeta)$, which is analytic at all points in a neighborhood of the origin of coordinates except, perhaps, the origin itself. Point $\zeta = 0$ is then the image of the point at infinity $z = \infty$, and each sequence of points $\{z_n\}$ that converges to the point at infinity has corresponding to it a sequence of points $\{\zeta_n = 1/z_n\}$ that converges to zero, and vice versa.

Depending on whether point $\zeta = 0$ is a regular point, a pole of the kth order, or an essential singularity of $f^*(\zeta)$, we call the point $z = \infty$ a regular point, a pole of the kth order, or an essential singularity. Since in all such cases $f^*(\zeta)$ can be expanded in a neighborhood of $\zeta = 0$ in Laurent series, which are

$$f^*(\zeta) = a_0 + a_{-1}\zeta + a_{-2}\zeta^2 + \ldots + a_{-n}\zeta^n + \ldots,$$

$$f^*(\zeta) = a_k\zeta^{-k} + \ldots + a_1\zeta^{-1} + a_0 + a_{-1}\zeta + \ldots \quad (a_k \neq 0),$$

$$f^*(\zeta) = \sum_{-\infty}^{+\infty} a_n\zeta^{-n}$$

(where in the last case there is an infinite number of terms with negative powers of ζ whose coefficients a_n are nonzero), the *function $f(z) = f^*(1/z)$ in a neighborhood of the point at infinity can be expanded, depending on whether the point is a regular point, a pole of the kth order, or an essential singularity, in the following Laurent series*:

$$f(z) = a_0 + a_{-1}z^{-1} + a_{-2}z^{-2} + \ldots + a_{-n}z^{-n} + \ldots,$$

$$f(z) = a_kz^k + \ldots + a_1z + a_0 + a_{-1}z^{-1} + \ldots \quad (a_k \neq 0),$$

$$f(z) = \sum_{-\infty}^{+\infty} a_nz^n$$

(where in the last case there is an infinite number of terms with *positive powers of z* whose coefficients are nonzero).

Thus, the relationship between the nature of a point with respect to a function and the corresponding Laurent expansion for this function is the same as in the case of a finite point, only the roles of terms with negative and positive powers are interchanged. The *principal part of a Laurent expansion in a neighborhood of the point at infinity* is the sum of terms with positive powers, and the *regular part* is the sum of terms with nonpositive powers.

We know how to distinguish between a regular point, a point, and an essential singularity at $\zeta = 0$ by not resorting to the re-

spective Laurent expansion. We only have to know which of the three possibilities is realized:

(1) $f^*(\zeta)$ is bounded in a neighborhood of the origin of coordinates;

(2) $f^*(\zeta) \to \infty$ as $\zeta \to 0$;

(3) there is neither a finite nor an infinite limit for $f^*(\zeta)$ as $\zeta \to 0$.

From the fact that $f(z) = f^*(\zeta)$ and $z = 1/\zeta$ it follows that the same criteria are valid in the case of the point at infinity, and $f(z)$ has a regular point, a pole, or an essential singularity at $z = \infty$ depending on whether the function is bounded in a neighborhood of the point at infinity, tends to infinity as z tends to infinity, or has no limit, finite or infinite, as z tends to infinity.

7.7
ENTIRE AND MEROMORPHIC FUNCTIONS

An entire function by definition (a function that is single-valued and analytic in the entire finite plane) has no singularities in the finite plane. Hence, it can have a singularity only at the point at infinity. Since $f(z)$ can be expanded everywhere into a convergent power series,

$$f(z) = a_0 + a_1 z + \dots + a_n z^n + \dots,$$

the series represents $f(z)$ in any neighborhood of the point at infinity and, therefore, coincides with the Laurent expansion of $f(z)$ in a neighborhood of that point. This implies that when point $z = \infty$ is regular for $f(z)$, the coefficients $a_1 = a_2 = \dots = a_n = \dots = 0$ and $f(z) \equiv a_0$. This also follows from Liouville's theorem: from the fact that $z = \infty$ is a regular point of $f(z)$ it follows that the modulus of $f(z)$ is bounded in a neighborhood of this point and, hence, is bounded in the entire plane, which implies that $f(z) \equiv$ const.

In the case where $z = \infty$ is a pole of the kth order of $f(z)$ we have

$$a_k \neq 0, \quad a_{k+1} = a_{k+2} = \dots = 0,$$

whence

$$f(z) \equiv a_0 + a_1 z + \dots + a_k z^k,$$

i.e. $f(z)$ is a polynomial (a rational entire function) of degree k.

Finally, when $z = \infty$ is an essential singularity of $f(z)$, there must be an infinite number of nonzero expansion coefficients in the Laurent expansion for $f(z)$. The function then is not a polynomial; it is called a *transcendental entire function*. Examples are e^z, $\sin z$, and $\cos z$.

Therefore, *an entire function may be at infinity regular, have a pole, or have an essential singularity*. Correspondingly, the entire function is a constant, a polynomial (whose degree coincides with the order of the pole), or a transcendental entire function.

A *meromorphic function* (from the Greek *meros* for part and *morphe* for form) is a function that in the finite plane can be represented by the quotient of two entire functions:

$$f(z)=\frac{g(z)}{h(z)}\quad (h(z)\not\equiv 0).$$

On the basis of the result obtained in Sec. 7.5 we conclude that in the finite plane a meromorphic function cannot have any singularities except poles. Namely, $f(z)$ has a pole at z_0 if and only if this point is a zero of the denominator ($h(z_0)=0$) and the numerator is nonzero at this point ($g(z_0)\neq 0$) or its zero is of a lower order than the order of the zero of the denominator. The order of the pole is equal to the difference between the orders of the zeros of the denominator and numerator. If in the entire finite plane there is only a finite number of poles of $f(z)$, there exists a neighborhood of point $z=\infty$ inside which there are no finite singular points of $f(z)$. Hence, just as in the case of an entire function, $f(z)$ can at ∞ be regular or have a pole or have an essential singularity. Suppose that $z_1, z_2, \ldots, z_m$ are the different poles of $f(z)$ and $\beta_1, \beta_2, \ldots, \beta_m$ are the orders of these poles. The Laurent expansion of $f(z)$ in a neighborhood of point z_j (a pole) is

$$f(z)=\frac{A^{(j)}_{-\beta_j}}{(z-z_j)^{\beta_j}}+\ldots+\frac{A^{(j)}_{-1}}{z-z_j}+A_0^{(j)}+A_1^{(j)}(z-z_j)+\ldots .$$

The principal part of this expansion is a rational function $G_j(z)$ with only one pole of the β_jth order at point z_j in the entire extended plane; at $z=\infty$ this function vanishes. Obviously, the difference $f(z)-G_j(z)$ has no singularity at $z=z_j$ but has poles at $z_1, \ldots, z_{j-1}, z_{j+1}, \ldots, z_m$. The principal parts of the $f(z)-G_j(z)$ coincide with the respective principal parts of $f(z)$; indeed, at each point z_k ($k\neq j$) the function $G_j(z)$ is regular and, hence, its expansion in a neighborhood of z_k contains only nonnegative powers of $z-z_k$, which do not affect any principal part. If we subtract the sum of the principal parts, $\sum\limits_{j=1}^{m} G_j(z)$, from $f(z)$, we arrive at a function $\varphi(z)=f(z)-\sum\limits_{j=1}^{m} G_j(z)$ that has no singular points in the finite plane and is, therefore, an entire function. Since $\lim\limits_{z\to\infty}[f(z)-\varphi(z)]=$ $=\lim\limits_{z\to\infty}\sum\limits_{j=1}^{m} G_j(z)=0$, the two functions $f(z)$ and $\varphi(z)$ behave in the same way at the point at infinity, i.e. they have there a regular point, a pole, or an essential singularity, and in the last two cases the principal parts of the Laurent expansions of $f(z)$ and $\varphi(z)$ at

$z = \infty$ coincide. Thus, a meromorphic function $f(z)$ can be represented in the form of the sum of an entire function $\varphi(z)$ and a rational function $\sum_{j=1}^{m} G_j(z)$:

$$f(z) = \varphi(z) + \sum_{j=1}^{m} G_j(z).$$

When at point $z = \infty$ a meromorphic function is regular or has a pole of the kth order, the entire function $\varphi(z)$ is, respectively, a constant ($\varphi(z) \equiv \lim_{z\to\infty} f(z) = A_0$) or a polynomial of degree k (i.e. $A_0 + A_1 z + \ldots + A_k z^k$, with $A_1 z + \ldots + A_k z^k$ the principal part of the Laurent expansion of $f(z)$ at $z = \infty$). If we denote this principal part by $G_0(z)$ and assume that $G_0(z) \equiv 0$ if the point at infinity is a regular point of $f(z)$, we obtain

$$f(z) = A_0 + \sum_{j=0}^{m} G_j(z). \tag{7.24}$$

This result can be stated in the form of a

Theorem. *If a meromorphic function $f(z)$ has only a finite number of poles in the finite plane and the point at infinity is a regular point of $f(z)$ or a pole, the function is rational; it can be represented in the form of a sum of its principal parts $G_j(z)$ with respect to all the finite poles ($j = 1, 2, \ldots, m$) and the point at infinity ($j = 0$) and a constant* $A_0 = \lim_{z\to\infty} [f(z) - \sum_{0}^{m} G_j(z)]$.

Note that this theorem establishes the existence of a partial-fraction expansion for any rational function. Indeed, if we write (7.24) in the expanded form, we have

$$f(z) = A_0 + A_1 z + \ldots + A_k z^k + \sum_{j=1}^{m} \left[\frac{A_{-\beta_j}^{(j)}}{(z - z_j)^{\beta_j}} + \ldots + \frac{A_{-1}^{(j)}}{z - z_j} \right],$$

which is exactly the partial-fraction expansion for $f(z)$.

Finally, let us consider the case where a meromorphic function $f(z)$ has an infinite number of poles. We can easily see that every closed circle $|z| \leqslant R < \infty$ can have only a finite number of poles inside it. Suppose the contrary is true. Then inside the circle we will find at least one point ζ that is a limit point for the poles. But such a point can be neither regular nor a pole of $f(z)$, since there is no neighborhood of this point in which $f(z)$ would be analytic everywhere except, perhaps, the point itself. Naturally, there can be no such (finite) point for a meromorphic function.

Thus, each circle $|z| \leqslant R$ contains only a finite number of poles of $f(z)$, in view of which for any value of R, a neighborhood $|z| > R$

of point ∞ contains an infinite number of poles of $f(z)$, which implies that in the case under consideration point ∞ is the limit point for the poles, i.e. a nonisolated singular point. Let us show that *all the poles of $f(z)$ can be numbered in such a way that they form a sequence with nondecreasing moduli*. To this end we divide the entire plane by means of circles centered at the origin of coordinates into zones: K_0 $(|z| \leqslant 1)$, K_1 $(1 < |z| \leqslant 2)$, K_2 $(2 < |z| \leqslant 3)$, ... Each of these zones contains only a finite number (zero, in particular) of poles of $f(z)$. Therefore, if we order the different poles so that they form a sequence with nondecreasing moduli,

$$0 \leqslant |z_0| \leqslant |z_1| \leqslant \ldots \leqslant |z_{n_j}|,$$

all lying in the closed circle $|z| \leqslant j$, we can continue the ordering by adding the poles that lie in the zone K_j adjacent to the circle:

$$|z_{n_j}| < |z_{n_j+1}| \leqslant \ldots \leqslant |z_{n_{j+1}}|.$$

Continuing this process indefinitely, we arrive at a sequence $\{z_n\}$ with all the poles of $f(z)$, where $|z_n| \leqslant |z_{n+1}|$.

7.8 THE MITTAG-LEFFLER THEOREM

We denote by $G_j(z)$ the principal part of the Laurent expansion of $f(z)$ in a neighborhood of point z_j:

$$G_j(z) = \frac{A^{(j)}_{-\beta_j}}{(z-z_j)^{\beta_j}} + \frac{A^{(j)}_{-\beta_j+1}}{(z-z_j)^{\beta_j-1}} + \ldots + \frac{A^{(j)}_{-1}}{z-z_j}.$$

We can try the same approach that we used in the case of a meromorphic function with a finite number of poles, namely, subtract the sum of all the principal parts from $f(z)$ so that the difference is a function without singularities in the finite plane, i.e. an entire function. But here we are dealing with an infinitude of principal parts, and there is no guarantee that the series $\sum\limits_{j=0}^{\infty} G_j(z)$ will converge. In 1877 the Swedish mathematician G. M. Mittag-Leffler overcame this difficulty by showing that one can find polynomials $P_j(z)$ such that the series $\sum\limits_{j=0}^{\infty} [G_j(z) - P_j(z)]$ is uniformly convergent in any circle $|z| < R$ (if from this circle we exclude the points z_0, z_1, . . . that are the poles of the series terms). If we suppose that such polynomials have been found, we find that the function $F(z) = \sum\limits_{j=0}^{\infty} [G_j(z) - P_j(z)]$ is analytic with poles at the same points z_j

where $f(z)$ has its poles, and the principal parts of $f(z)$ and $F(z)$ at each point z_j are the same rational functions $G_j(z)$. Consequently, $f(z) - F(z) = \varphi(z)$ is an entire function, which yields

$$f(z)=\varphi(z)+\sum_{j=0}^{\infty}[G_j(z)-P_j(z)]$$

$$=\varphi(z)+\sum_{j=0}^{\infty}\left[\frac{A_{-\beta_j}^{(j)}}{(z-z_j)^{\beta_j}}+\ldots+\frac{A_{-1}^{(j)}}{z-z_j}-(C_0^{(j)}+\ldots+C_{\alpha_j}^{(j)}z^{\alpha_j})\right]. \tag{7.25}$$

This is the so-called *Mittag-Leffler expansion* of a meromorphic function $f(z)$. It can be considered the *partial-fraction expansion of a meromorphic function.*

Let us prove the Mittag-Leffler theorem in its general form. Given a sequence of different complex numbers $\{z_n\}$, $|z_{n+1}| \geqslant |z_n|$, $z_n \to \infty$, and a sequence of rational functions $\{G_n(z)\}$,

$$G_n(z)=\sum_{j=1}^{\beta_n}\frac{A_{-j}^{(n)}}{(z-z_n)^j}, \quad \beta_n \geqslant 1, \ n=0, 1, 2, \ldots .$$

We will prove that there exists an analytic function $f(z)$ with poles at the given points z_n and only at these points and with the principal part of $f(z)$ at each point z_n that coincides with the corresponding function $G_n(z)$.

Let us consider a convergent number series $\sum_{1}^{\infty} \varepsilon_n$ with positive terms ($\varepsilon_n = 1/n^2$, for instance). If $z_n \neq 0$ (this condition is met, obviously, for $n \geqslant 1$), $G_n(z)$ is analytic in the circle $|z| < |z_n|$ and, hence, can be expanded in this circle in a power series in z:

$$G_n(z)=\sum_{j=0}^{\infty} c_j^{(n)}z^j, \quad |z|<|z_n|.$$

Since in the circle $|z| \leqslant |z_n|/2$ this series is uniformly convergent, there exists a nonnegative α_n such that in the above-mentioned closed circle the inequality

$$\left|G_n(z)-\sum_{0}^{\alpha_n} c_j^{(n)}z^j\right|<\varepsilon_n$$

is valid.

We set

$$P_n(z)=\sum_{0}^{\alpha_n} c_j^{(n)}z^j, \quad n>0, \text{ and } P_0(z)\equiv 0$$

and consider the series

$$\sum_{0}^{\infty} [G_n(z) - P_n(z)].$$

We take an arbitrarily large positive number R. Since $z_n \to \infty$ as $n \to \infty$, we find that $|z_n| > 2R$ when $n > N(R)$. Therefore, in the circle $|z| < R < |z_n|/2$ we have

$$|G_n(z) - P_n(z)| < \varepsilon_n, \quad n > N(R),$$

and, hence, $\sum_{N(R)+1}^{\infty} [G_n(z) - P_n(z)]$ is analytic inside it. This implies that

$$f(z) = \sum_{0}^{\infty} [G_n(z) - P_n(z)]$$

$$= \sum_{0}^{N(R)} [G_n(z) - P_n(z)] + \sum_{N(R)+1}^{\infty} [G_n(z) - P_n(z)]$$

is also analytic in the same circle except at the poles belonging to the sequence $\{z_n\}$. The principal part of $f(z)$ at each of the poles z_n coincides with the given function $G_n(z)$. Since R is arbitrary, it follows, finally, that $f(z)$ is a function analytic in the plane except at the poles and thus satisfies the hypothesis of the theorem.

Here is a simple example of an expansion of type (7.25), the validity of which will be proved in Sec. 8.5:

$$\cot z = \frac{1}{z} + \sum_{j=1}^{\infty} \left(\frac{1}{z - z_j} + \frac{1}{z_j} \right) = \frac{1}{z} + \sum_{k=1}^{\infty} \left(\frac{1}{z - k\pi} + \frac{1}{z + k\pi} \right),$$

with $z_0 = 0$, $z_{2j-1} = j\pi$, $z_{2j} = -j\pi$ $(j = 1, 2, 3, \ldots)$, $1/(z - z_j)$ the principal part $G_j(z)$ of the Laurent expansion of $\cot z$ in a neighborhood of point z_j, and $1/z_j$ the simplest polynomial $P_j(z)$ (in this case of the zeroth degree) that must be added to $G_j(z)$ to ensure convergence. Finally, the entire function $\varphi(z)$ from the general formula (7.25) is in this particular case identically zero.

7.9
PRODUCT EXPANSION OF ENTIRE FUNCTIONS

Here we will show that for every entire function $f(z)$ that has zeros there exists a product expansion similar to that for a polynomial. To emphasize the analogy we write the product expansion for a polynomial in a form somewhat different from the commonly used. We assume that $P(z)$ is a polynomial, $z_1, \ldots, z_n$ its zeros differing

from the origin of coordinates (there may be equal zeros among these, which correspond to multiple roots), and $z = 0$ a zero of $P(z)$ of the λth order (if $P(0) \neq 0$, we put $\lambda = 0$). Then we can write

$$P(z) = Cz^{\lambda}\left(1+\frac{z}{z_1}\right)\ \ldots\ \left(1-\frac{z}{z_n}\right) = Cz^{\lambda}\prod_{1}^{n}\left(1-\frac{z}{z_j}\right).$$

For comparison, let us take the entire function $\sin z$, which has simple zeros at $0, \pm\pi, \pm 2\pi, \ldots, \pm n\pi, \ldots$. If by putting $z_{2j-1} = j\pi$ and $z_{2j} = -j\pi$ we order the zeros of $\sin z$ that differ from the origin of coordinates in a way such that their moduli do not decrease, we can prove that

$$\sin z = z\prod_{j=1}^{\infty}\left(1-\frac{z}{z_j}\right) = z\prod_{k=1}^{\infty}\left(1-\frac{z^2}{k^2\pi^2}\right). \tag{7.26}$$

This formula appears to be very similar to that for polynomials. We will prove its validity in Sec. 8.5.

In the general case, however, it is not possible to obtain such a simple formula. The fact is that if $f(z)$ is an entire function with the following zeros: 0 (of the λth order), $z_1, z_2, \ldots, z_n, \ldots$ $\ldots, (|z_n| \leqslant |z_{n+1}|)$, the product $z^{\lambda}\prod_{j=1}^{\infty}\left(1-\frac{z}{z_j}\right)$ may diverge. To overcome this difficulty, Weierstrass introduced into this product additional factors of the type $e^{\frac{z}{z_j}+\frac{z^2}{2z_j^2}+\ldots+\frac{z^{k_j}}{k_j z_j^{k_j}}}$, which never vanish but ensure the convergence of the infinite product. If we select the k_j in a way such that the product $z^{\lambda}\prod_{j=1}^{\infty}\left(1-\frac{z}{z_j}\right)\times$ $\times\, e^{\frac{z}{z_j}+\ldots+\frac{z^{k_j}}{k_j z_j^{k_j}}}$ converges uniformly in every closed circle $|z| \leqslant R$, the product is an entire function $F(z)$ with the same zeros as $f(z)$. From this we can prove that $f(z)$ either coincides with $F(z)$ or differs from $F(z)$ by a factor of the type $e^{g(z)}$, where $g(z)$ is also an entire function (the factor is an entire factor that does not vanish anywhere). For $f(z)$ we then have the *Weierstrass formula* (1876)

$$f(z) = e^{g(z)}z^{\lambda}\prod_{j=1}^{\infty}\left(1-\frac{z}{z_j}\right)e^{\frac{z}{z_j}+\ldots+\frac{z^{k_j}}{k_j z_j^{k_j}}}. \tag{7.27}$$

We now prove the validity of this formula.

Lemma. *If two entire functions $f(z) \not\equiv 0$ and $F(z) \not\equiv 0$ have the same zeros (of the same orders), then*

$$f(z) = e^{g(z)} F(z),$$

where $g(z)$ is an entire function (a constant, for one).

Indeed, the quotient $\varphi(z) = f(z)/F(z)$ can have singularities (poles) only at points where $F(z)$ has its zeros. But each zero of $F(z)$ is a zero of $f(z)$ of the same order, whereby $\varphi(z)$ has no poles in the finite plane, i.e. it is an entire function. The same reasoning shows that $\varphi(z)$ has no zeros in the finite plane, whereby $h(z) = \varphi'(z)/\varphi(z)$ is an entire function (Sec. 7.5, item (a)). If we integrate $h(z)$ from zero to an arbitrary z, we again arrive at an entire function

$$g_1(z) = \int_0^z h(z)\,dz = \int_0^z \frac{\varphi'(z)}{\varphi(z)}\,dz = \ln\frac{\varphi(z)}{\varphi(0)} + 2k\pi i,$$

whence

$$\varphi(z) = \varphi(0)\,e^{g_1(z)} = e^{g_1(z)+\ln\varphi(0)} = e^{g(z)},$$

where $g(z) = g_1(z) + \ln\varphi(0)$ is also an entire function. Thus

$$f(z) = \varphi(z)\,F(z) = e^{g(z)} F(z).$$

We have therefore established that any entire function $\varphi(z)$ that has no zeros can be represented in the form $\varphi(z) = e^{g(z)}$, with $g(z)$ also an entire function.

Suppose $f(z)$ is an entire function with a finite number of zeros: $\underbrace{0, \ldots, 0}_{\lambda}, z_1, z_2, \ldots, z_n$. Since $F(z) = z^\lambda\left(1-\frac{z}{z_1}\right)\ldots\left(1-\frac{z}{z_n}\right)$ has the same zeros, by the above lemma

$$f(z) = e^{g(z)} z^\lambda \prod_{j=1}^{n}\left(1-\frac{z}{z_j}\right).$$

To obtain a similar formula for an entire function with an infinite number of zeros, we must build at least one such function with preassigned zeros. Let us prove that such a function always exists.

Theorem. *For any sequence of complex numbers $\{z_n\}$ that converges to ∞ and whose terms differ from the origin of coordinates and are ordered in a way such that their moduli do not decrease ($|z_n| \leqslant |z_{n+1}|$) we can build an entire function $f(z)$ with zeros that coincide with the z_n.*

Remark. Such a sequence may contain equal terms (placed one after another); if $z_{n_0+1} = z_{n_0+2} = \ldots = z_{n_0+\alpha} = a$, with the other terms differing from a, the number a is a zero of $f(z)$ of the αth order.

Proof. Suppose that $\{k_n\}$ is a sequence of nonnegative integers such that $\sum_{n=1}^{\infty}\left(\frac{R}{|z_n|}\right)^{k_n+1}$ converges for any $R \geqslant 0$. For any sequence $\{z_n\}$ we can put $k_n = [\ln n]$, where $[\ln n]$ is the integral part of $\ln n$. Indeed, if $|z_n|/R > e^2$ for $n > N_0$, then $\left(\frac{|z_n|}{R}\right)^{k_n+1} > e^{2([\ln n]+1)} > e^{2\ln n} = n^2$, whence $\left(\frac{R}{|z_n|}\right)^{k_n+1} < \frac{1}{n^2}$. But in some cases we can take equal numbers for the k_n. For instance, if $|z_n| = n$, we may put $k_n = 1$; if $z_n = n^2$, it suffices to put $k_n = 0$ $(n = 1, 2, \ldots)$. Let us show that the sought for function can be written in the form of an infinite product $\prod_1^{\infty}\left(1-\frac{z}{z_j}\right) \times e^{\frac{z}{z_j}+\ldots+\frac{z^{k_j}}{k_j z_j^{k_j}}}$ (if $k_j = 0$, we must assume that the corresponding exponent is zero). Suppose R is an arbitrary positive number; by $N(R)$ we denote a positive integer such that $|z_n| > 2R$ if $n > N(R)$. Assuming that $n > N(R)$, we write the product $\prod_{j=1}^{n}$ in the form

$$\prod_{j=1}^{n}\left(1-\frac{z}{z_j}\right)e^{\frac{z}{z_j}+\ldots+\frac{z^{k_j}}{k_j z_j^{k_j}}} = \prod_{j=1}^{N(R)} \times \prod_{N(R)+1}^{n}. \tag{7.28}$$

The first product is an entire function with the given zeros z_1, $z_2, \ldots, z_{N(R)}$; hence, in the circle $|z| \leqslant 2R$ the zeros of this product satisfy the hypothesis of the theorem. We write the second product in the form

$$\prod_{N(R)+1}^{n} = \prod_{N(R)+1}^{n} \exp\left\{\ln\left(1-\frac{z}{z_j}\right)+\frac{z}{z_j}+\ldots+\frac{z^{k_j}}{k_j z_j^{k_j}}\right\}$$
$$= \exp \sum_{N(R)+1}^{n}\left\{\ln\left(1-\frac{z}{z_j}\right)+\frac{z}{z_j}+\ldots+\frac{z^{k_j}}{k_j z_j^{k_j}}\right\}.$$

Since $|z| \leqslant R$ and $|z_j| > 2R$, we find that $|z/z_j| < 1/2$, whence

$$\left|\ln\left(1-\frac{z}{z_j}\right)+\frac{z}{z_j}+\ldots+\frac{z^{k_j}}{k_j z_j^{k_j}}\right|$$
$$= \left|\frac{z^{k_j+1}}{(k_j+1)\, z_j^{k_j+1}}+\frac{z^{k_j+2}}{(k_j+2)\, z_j^{k_j+2}}+\ldots\right|$$

$$\leqslant \frac{|z|^{k_j+1}}{|z_j|^{k_j+1}}+\frac{|z|^{k_j+2}}{|z_j|^{k_j+2}}+\ldots \leqslant \frac{R^{k_j+1}}{|z_j|^{k_j+1}}\left(1+\frac{R}{|z_j|}+\frac{R^2}{|z_j|^2}+\ldots\right)$$

$$=\frac{R^{k_j+1}}{|z_j|^{k_j+1}}\frac{1}{1-\frac{R}{|z_j|}}<\frac{2R^{k_j+1}}{|z_j|^{k_j+1}} \quad (j>N(R)).$$

From this inequality and from the fact that the series $\sum\limits_{1}^{\infty}\left(\frac{R}{|z_j|}\right)^{k_j+1}$ converges it follows that the series $\sum\limits_{N(R)+1}^{\infty}\left\{\ln\left(1-\frac{z}{z_j}\right)+\frac{z}{z_j}+\ldots+\frac{z^{k_j}}{k_j z_j^{k_j}}\right\}$ converges absolutely and uniformly in the circle $|z|<R$; whence the sum of the series, $\varphi_R(z)$, is single-valued and analytic in this circle.

Going over to the limit as $n\to\infty$ in (7.28), we obtain

$$\lim_{n\to\infty}\prod_{j=1}^{n}\left(1-\frac{z}{z_j}\right)e^{\frac{z}{z_j}+\ldots+\frac{z^{k_j}}{k_j z_j^{k_j}}}=\prod_{j=1}^{\infty}=\prod_{j=1}^{N(R)}\times e^{\varphi_R(z)}. \tag{7.29}$$

This limit is a function analytic in the circle $|z|<R$ and possessing the given zeros. Since this conclusion is valid for a circle of an arbitrary large radius R, the infinite product $\prod\limits_{j=1}^{\infty}$ converges in the entire finite plane to a function $f(z)$ that is analytic everywhere, i.e. an entire function, with the preassigned zeros $\{z_n\}$:

$$f(z)=\prod_{j=1}^{\infty}\left(1-\frac{z}{z_j}\right)e^{\frac{z}{z_j}+\ldots+\frac{z^{k_j}}{k_j z_j^{k_j}}}. \tag{7.30}$$

Examples. (a) $z_n=n^2$; here we may put $k_n=0$. The simplest entire function with these zeros is

$$f(z)=\prod_{1}^{\infty}\left(1-\frac{z}{n^2}\right).$$

(b) $z_n=n$; here we may put $k_n=1$. We obtain

$$f(z)=\prod_{1}^{\infty}\left(1-\frac{z}{n}\right)e^{\frac{z}{n}}.$$

(c) Suppose the sequence $\{z_n\}$ is 1, 2, 2, 3, 3, 3, 4, 4, 4, 4, 5, 5, 5, 5, 5, . . .; here we take $k_n=2$. The corresponding infinite

product can be written as

$$f(z)=(1-z)\,e^{z+\frac{z^2}{2}}\left[\left(1-\frac{z}{2}\right)e^{\frac{z}{2}+\frac{z^2}{8}}\right]^2\left[\left(1-\frac{z}{3}\right)e^{\frac{z}{3}+\frac{z^2}{18}}\right]^3\ldots.$$

Suppose now that $f(z)\not\equiv 0$ is an arbitrary entire function with an infinitude of zeros. Noting that each closed circle $|z|\leqslant R<\infty$ can contain only a finite number of zeros, we find that just as in the case of the poles of a meromorphic function (Sec. 7.7) all the zeros can be ordered in a sequence with nondecreasing moduli:

$$\underbrace{0,\ 0,\ \ldots,\ 0}_{\lambda},\ z_1,\ z_2,\ \ldots,\ z_n,\ \ldots,\qquad \lim z_n=\infty.$$

If $k_n\geqslant 0$ are integers such that the series $\sum\limits_1^{\infty}\left(\frac{R}{|z_n|}\right)^{k_n+1}$ converges for any R, then, by the theorem just proved, the function

$$F(z)=z^{\lambda}\prod_1^{\infty}\left(1-\frac{z}{z_n}\right)e^{\frac{z}{z_n}+\ldots+\frac{z^{k_n}}{k_n z_n^{k_n}}}$$

is an entire function with the same zeros as $f(z)$. Hence,

$$f(z)=e^{g(z)}z^{\lambda}\prod_1^{\infty}\left(1-\frac{z}{z_n}\right)e^{\frac{z}{z_n}+\ldots+\frac{z^{k_n}}{k_n z_n^{k_n}}},$$

where $g(z)$ is an entire function (it may even be equal to zero). This is the sought for Weierstrass formula.

The main theorem of this section enables us to prove that *every function $\Phi(z)$ that is single-valued and analytic in the finite plane, except at some points where it has its poles, is a meromorphic function, i.e. can be represented by a quotient of two entire functions.* Indeed, let us build an entire function $F(z)$ whose zeros are at points where $\Phi(z)$ has its poles, and the order of each zero is assumed to be equal to the order of the corresponding pole. Then the product $F(z)\Phi(z)=H(z)$ is, as we can easily see, an entire function, which implies that

$$\Phi(z)=\frac{H(z)}{F(z)}$$

is a meromorphic function.

We can therefore say that *the two classes of analytic functions, namely, (a) functions with no other singularities in the finite plane except poles and (b) functions that can be represented by a quotient of two entire functions, are equivalent.* Indeed, we have just established that each function of class (a) belongs to class (b). Conversely, a function of class (b) cannot have any other finite singular points except

poles (Sec. 7.7), from which follows the above statement. We can now give the following definition of a meromorphic function (which is equivalent to the one given before): *a single-valued analytic function $f(z)$ is called meromorphic if it has no singularities in the finite plane other than poles.*

7.10

THE GAMMA FUNCTION

The gamma function $\Gamma(z)$ is the simplest and most important of the infinitude of meromorphic functions. It generalizes the idea of the n factorial, $n!$, to the case of arbitrary complex numbers z. For historical reasons this function, which was introduced by L. Euler*, is defined in a way such that the $n!$ is obtained if we put $z = n + 1$, i.e. $\Gamma(n + 1) = n!$. Hence, the common relation

$$(n - 1)!\, n = n!$$

in the case of the gamma function is written thus:

$$n\Gamma(n) = \Gamma(n + 1).$$

Another relationship for the gamma function is

$$\Gamma(1) = 0! = 1.$$

Let us now build this function. We first require that

$$z\Gamma(z) = \Gamma(z + 1)$$

for all complex values of z. We will therefore start in our definition of the gamma function from the functional relation

$$zf(z) = f(z + 1), \quad f(1) = 1. \tag{7.31}$$

But this is not sufficient to determine the gamma function completely. Indeed, if $f_0(z)$ is a meromorphic function for which

$$zf_0(z) = f_0(z + 1), \quad f_0(1) = 1,$$

then for the ratio $\varphi(z) = f(z)/f_0(z)$ we find that $\varphi(z) = \varphi(z + 1)$ and $\varphi(1) = 1$, i.e. $\varphi(z)$ is a meromorphic function with a period equal to unity and which is unity when $z = 1$.

With such a function any of the functions $f(z)$ that satisfy (7.31) can be written as

$$f(z) = \varphi(z)\, f_0(z).$$

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* It is interesting to note that the Russian mathematician N. I. Lobachevsky studied the gamma function and obtained some remarkable results, which were new for his time. See the paper by G. L. Lunts "The works of N. I. Lobachevsky in mathematical analysis", in *Studies in the History of Mathematics* [in Russian], issue 2, Eds. G. F. Rybkin and A. P. Yushkevich, Gostekhteorizdat, Moscow-Leningrad, 1949.

If we substitute z, $z+1$, . . ., $z+n-1$ (with n a positive integer) for z in (7.31) and multiply the obtained relationships, we find that

$$z(z+1)\ldots(z+n-1)f(z)=f(z+n). \tag{7.32}$$

Whence, for $z=1$ we have

$$f(n+1)=n! \tag{7.33}$$

i.e. at points 2, 3, . . ., $n+1$, . . . the values of $f(z)$ correspond to 1!, 2!, . . ., $n!$, . . ., respectively (this is true for $z=1$ since $f(1)=1=0!$). If in (7.32) $z\to-(n-1)=-m$ ($m=0, 1, 2, \ldots$), we have

$$\lim_{z\to-m}(z+m)f(z)=\frac{f(1)}{(-1)^m\,m!}=\frac{(-1)^m}{m!}. \tag{7.34}$$

Hence, $f(z)$ must have simple poles at all points where $z=-m$ ($m=0, 1, 2, \ldots$). The principal part of the Laurent expansion of $f(z)$ at $z=-m$ is equal to $(-1)^m/m!\,(z+m)$.

We subject $f(z)$ to the following additional condition:

$$f(z) \text{ has no poles except at } z=-m \quad (m=0, 1, 2, \ldots) \text{ and has no zeros.} \tag{7.35}$$

If a function $f_0(z)$ satisfies two conditions, (7.31) and (7.35), then the function $f_0(z)\tan(i+\pi z)/\tan i$, for instance, satisfies (7.31) but does not satisfy (7.35) since it has an infinitude of imaginary zero points and poles. However, along with $f_0(z)$ the same conditions (7.31) and (7.35) are satisfied by any function $\varphi(z)f_0(z)$, where $\varphi(z)$ is an entire function with a period equal to unity, equal to unity at $z=0$, and with no zeros. Thus, there is still an ambiguity in the definition of the gamma function even after condition (7.35) is introduced. But by introducing this restriction we can at least assert that

$$F(z)=\frac{1}{f(z)} \tag{7.36}$$

is an entire function with simple zeros at $z=-m$ ($m=0, 1, 2, \ldots$) and only such zeros. It can therefore be written thus:

$$F(z)=e^{g(z)}z\prod_{1}^{\infty}\left(1+\frac{z}{m}\right)e^{-\frac{z}{m}}, \tag{7.37}$$

where $g(z)$ is an entire function $\Big($in choosing the factor $e^{-z/m}$ that ensures the convergence of the product we employ the fact that $\sum\limits_{1}^{\infty}\frac{1}{m^2}$ converges$\Big)$. Hence, each meromorphic function $f(z)$ that satis-

fies conditions (7.31) and (7.35) must have the form

$$f(z)=e^{-g(z)}\frac{1}{z\prod\limits_{1}^{\infty}\left(1+\frac{z}{m}\right)e^{-\frac{z}{m}}}. \tag{7.38}$$

Obviously, it satisfies condition (7.35) with any entire function $g(z)$. To study condition (7.31), we write (7.38) in the form

$$f(z)=\lim_{n\to\infty}\frac{e^{-g(z)}}{z\prod\limits_{1}^{n}\left(1+\frac{z}{m}\right)e^{-\frac{z}{m}}}$$

$$=\lim_{n\to\infty}\frac{n!\exp\left[-g(z)+\sum\limits_{1}^{n}\frac{z}{m}\right]}{z(z+1)\dots(z+n)}. \tag{7.38'}$$

Assuming, for the sake of brevity, that

$$\frac{n!\exp\left[-g(z)+\sum\limits_{1}^{n}\frac{z}{m}\right]}{z(z+1)\dots(z+n)}=f_n(z),$$

we obtain

$$\frac{zf(z)}{f(z+1)}=\lim_{n\to\infty}\frac{zf_n(z)}{f_n(z+1)}$$

$$=\lim_{n\to\infty}(z+n+1)\exp\left[-g(z)+g(z+1)-\sum_{1}^{n}\frac{1}{m}\right]$$

$$=\lim_{n\to\infty}\left(1+\frac{z+1}{n}\right)\exp\left[-g(z)+g(z+1)\right.$$

$$\left.-\left(\sum_{1}^{n}\frac{1}{m}-\ln n\right)\right]=\exp\left[-g(z)+g(z+1)-C\right],$$

where C is the well-known Euler constant

$$\lim_{n\to\infty}\left(\sum_{1}^{n}\frac{1}{m}-\ln n\right)=0.5772\dots .$$

Thus, we satisfy the functional relationship (7.31) if we subject the entire function $g(z)$ to the following restriction:

$$g(z+1)-g(z)=C+2k\pi i \quad (k \text{ an integer}).$$

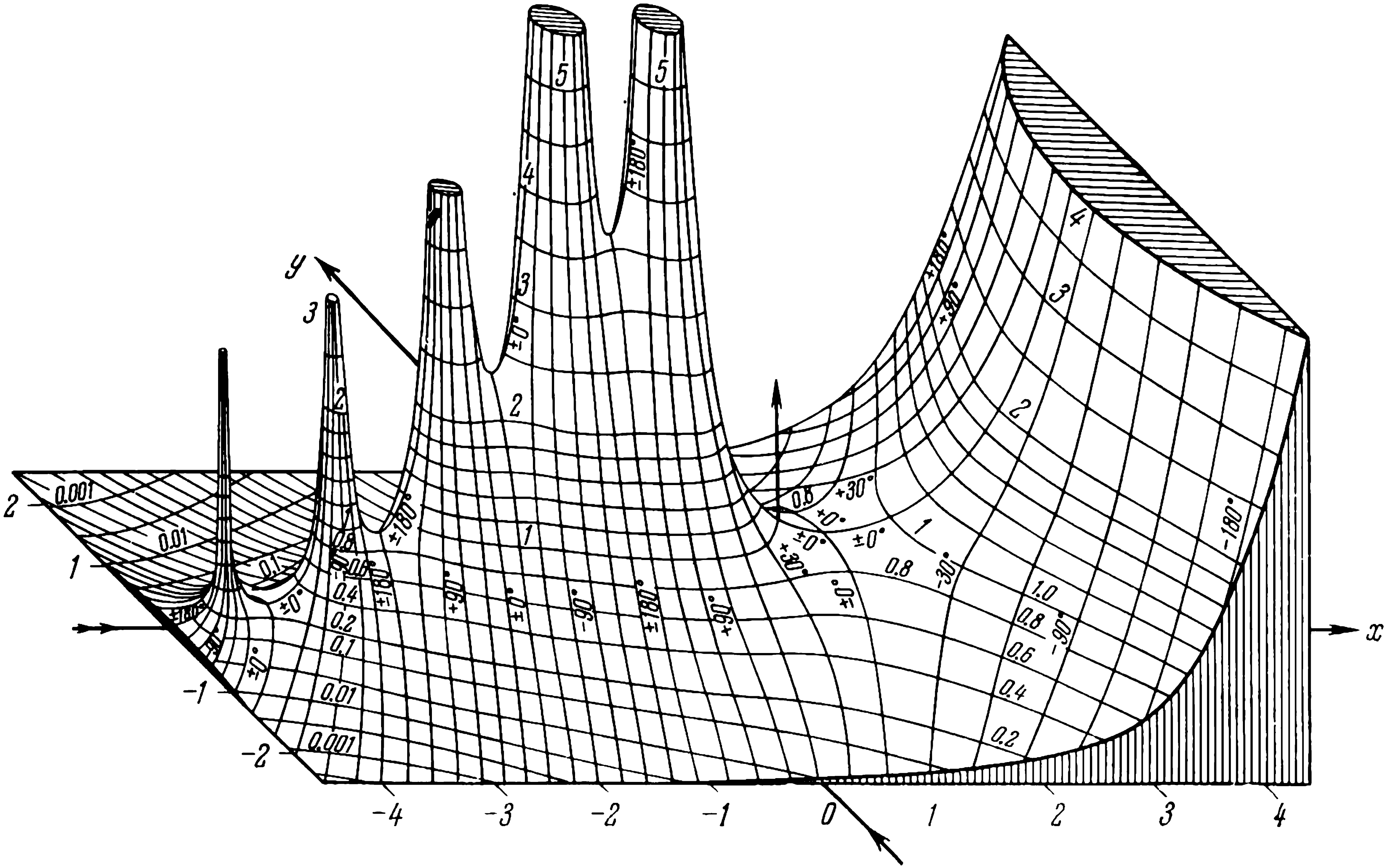

Fig. 51

In addition, the condition $f(1) = 1$ yields

$$f(1) = \lim_{n\to\infty} f_n(1) = \lim_{n\to\infty} \frac{\exp\left[-g(1) + \sum_{1}^{n} \frac{1}{m} - \ln n\right]}{1 + \frac{1}{n}}$$

$$= \exp[-g(1) + C],$$

whence

$$g(1) = C + 2l\pi i \quad (l \text{ an integer}).$$

The simplest entire function that satisfies the above conditions is the linear function

$$g_0(z) = Cz.$$

The definition of the gamma function will be complete if we choose $g(z)$ in the form of a linear function. If we denote the gamma function by $\Gamma(z)$, we have, by formula (7.38),

$$\Gamma(z) = e^{-Cz} \frac{1}{z \prod_{1}^{\infty} \left(1 + \frac{z}{m}\right) e^{-\frac{z}{m}}}. \tag{7.39}$$

For the entire function $1/\Gamma(z)$ we then have

$$\frac{1}{\Gamma(z)} = e^{Cz} z \prod_{1}^{\infty} \left(1 + \frac{z}{m}\right) e^{-\frac{z}{m}} \tag{7.40}$$

which is known as the *Weierstrass canonical form*.

An idea of the behavior of $\Gamma(z)$ and $1/\Gamma(z)$ is given by the modular surfaces of these functions, which are depicted in Figs. 51 and 52.

There is an important relationship connecting $\Gamma(z)$ and $\sin \pi z$, which follows from (7.40) and (7.26), namely

$$\frac{1}{\Gamma(z)\Gamma(-z)} = -z^2 \prod_{1}^{\infty} \left(1 - \frac{z^2}{m^2}\right)$$

$$= -\frac{z}{\pi}\left[\pi z \prod_{1}^{\infty} \left(1 - \frac{\pi^2 z^2}{m^2 \pi^2}\right)\right] = -\frac{z \sin \pi z}{\pi}, \tag{7.41}$$

or

$$\frac{1}{\Gamma(z)[-z\Gamma(-z)]} = \frac{\sin \pi z}{\pi}.$$

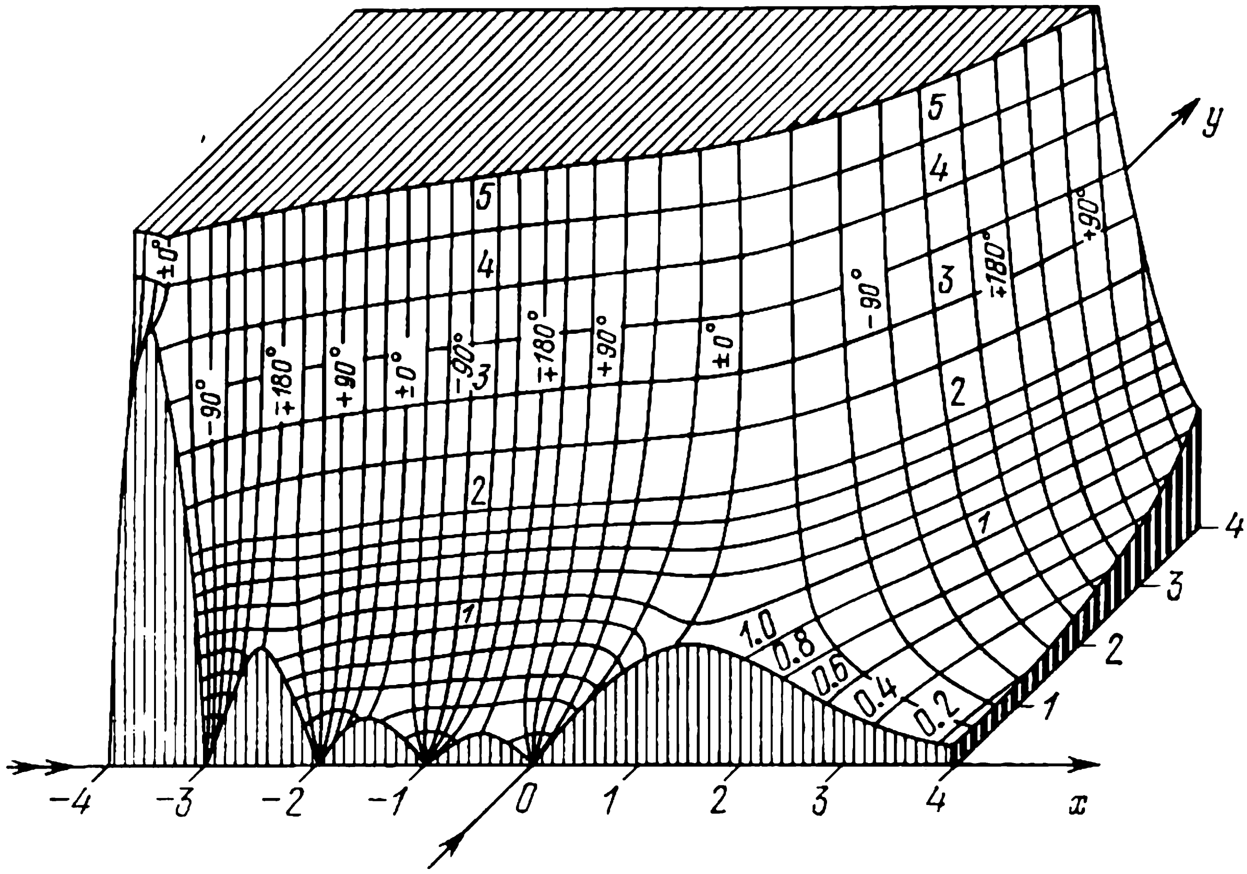

Fig. 52

But according to (7.31), $-z\Gamma(-z) = \Gamma(1-z)$, whence

$$\frac{1}{\Gamma(z)\,\Gamma(1-z)} = \frac{\sin \pi z}{\pi},$$

or

$$\Gamma(z)\,\Gamma(1-z) = \frac{\pi}{\sin \pi z}. \tag{7.41'}$$

This, for one, implies that

$$\left[\Gamma\left(\frac{1}{2}\right)\right]^2 = \pi,$$

and since according to (7.39) $\Gamma(1/2) > 0$, we find that

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}.$$

For $g(z) = Cz$ formula (7.38′) yields the following representation for the gamma function:

$$\Gamma(z) = \lim_{n\to\infty} \frac{n!\exp\left[\left(\sum\limits_{1}^{n} \frac{1}{m} - C\right) z\right]}{z(z+1)\dots(z+n)}$$

$$= \lim_{n\to\infty} \frac{n!\exp\left\{\left[\left(\sum\limits_{1}^{n} \frac{1}{m} - \ln n - C\right) + \ln n\right] z\right\}}{z(z+1)\dots(z+n)},$$

and since

$$\lim \left(\sum_{1}^{n} \frac{1}{m} - \ln n - C \right) = 0 \text{ and } \exp(z \ln n) = n^z,$$

we find that

$$\Gamma(z) = \lim_{n \to \infty} \frac{n!\, n^z}{z(z+1) \ldots (z+n)}. \tag{7.42}$$

This formula was first derived by L. Euler and should be named after him, but it is more commonly known as the *Gauss formula.*

7.11

AN INTEGRAL REPRESENTATION OF THE GAMMA FUNCTION

We will first prove that when Re $(z) > 0$,

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1}\, dt, \tag{7.43}$$

where integration is carried out along the positive half of the real axis. This formula was also discovered by Euler and is known as *Euler's integral of the second kind.*

We consider the integral

$$F(z) = \int_0^\infty e^{-t} t^{z-1}\, dt, \tag{7.44}$$

where by t^{z-1} we denote exp $[(z - 1) \ln t]$. Since $| e^{-t} t^{z-1} | = e^{-t} t^{x-1}$, the above integral is absolutely convergent for every z that belongs to the domain D: x = Re $(z) > 0$.

We prove the auxiliary relation

$$\Gamma(z) = \lim_{n \to \infty} \int_0^n \left(1 - \frac{t}{n}\right)^n t^{z-1}\, dt. \tag{7.45}$$

Changing the variable t to $n\tau$, we have

$$F_n(z) = \int_0^n \left(1 - \frac{t}{n}\right)^n t^{z-1}\, dt = n^z \int_0^1 (1-\tau)^n \tau^{z-1}\, d\tau,$$

or, after we integrate n times by parts,

$$F_n(z) = \frac{n!\, n^z}{z(z+1) \ldots (z+n-1)} \int_0^1 \tau^{z+n-1}\, d\tau = \frac{n!\, n^z}{z(z+1) \ldots (z+n)}.$$

But this implies that (7.45) coincides with (7.42) (for $\operatorname{Re}(z) > 0$) and, therefore, is valid. Now we must prove that the same formula is valid for $F(z)$:

$$F(z) = \lim_{n\to\infty} F_n(z), \quad z \in D, \tag{7.46}$$

or that

$$\lim_{n\to\infty} \int_0^n \left[e^{-t} - \left(1 - \frac{t}{n}\right)^n\right] t^{z-1}\,dt = 0, \quad z \in D,$$

$$\left(\text{since } F(z) = \lim_{n\to\infty} \int_0^n e^{-t} t^{z-1}\,dt\right).$$

Noting that, for $|t| < n$,

$$1 + \frac{t}{n} \leqslant e^{\frac{t}{n}} \leqslant \frac{1}{1 - \frac{t}{n}},$$

we have

$$\left(1 + \frac{t}{n}\right)^n \leqslant e^t \quad \text{and} \quad \left(1 - \frac{t}{n}\right)^n \leqslant e^{-t};$$

hence

$$0 \leqslant e^{-t} - \left(1 - \frac{t}{n}\right)^n = e^{-t}\left[1 - e^t\left(1 - \frac{t}{n}\right)^n\right] \leqslant e^{-t}\left[1 - \left(1 - \frac{t^2}{n^2}\right)^n\right]$$

$$= e^{-t}\frac{t^2}{n^2}\left[1 + \left(1 - \frac{t^2}{n^2}\right) + \ldots + \left(1 - \frac{t^2}{n^2}\right)^{n-1}\right] \leqslant \frac{t^2 e^{-t}}{n}.$$

Therefore,

$$\left|\int_0^n \left[e^{-t} - \left(1 - \frac{t}{n}\right)^n\right] t^{z-1}\,dt\right| < \frac{1}{n}\int_0^n e^{-t} t^{x+1}\,dt < \frac{1}{n}\int_0^\infty e^{-t} t^{x+1}\,dt,$$

from which (7.46) follows. Thus, we have proved the validity of (7.43).

Let us write (7.43) in the following form:

$$\Gamma(z) = \int_0^1 e^{-t} t^{z-1}\,dt + \int_1^\infty e^{-t} t^{z-1}\,dt. \tag{7.47}$$

The first integral on the right-hand side is a function analytic in the half-plane D (see Sec. 6.10). Substituting for e^{-t} its power series expansion and integrating termwise, we have

$$\varphi(z) = \int_0^1 e^{-t} t^{z-1}\,dt = \sum_0^\infty \frac{(-1)^n}{n!} \int_0^1 t^{n+z-1}\,dt = \sum_0^\infty \frac{(-1)^n}{n!\,(n+z)}. \tag{7.48}$$

We have this formula for the case where Re $(z) > 0$. But the series converges absolutely and uniformly in every bounded domain not including points $0, -1, -2, \ldots$. For this reason it represents a function $\varphi(z)$ that is single-valued and analytic in the entire finite plane except at the points $0, -1, -2, \ldots$, at which $\varphi(z)$ has simple poles. Hence, $\varphi(z)$ is a meromorphic function. The principal part of the Laurent expansion of $\varphi(z)$ at each pole coincides, obviously, with the principal part of $\Gamma(z)$ at the same pole. Therefore, the difference $\Gamma(z) - \varphi(z)$ is an entire function. From (7.47) it then follows that in D this difference is exactly $\int_1^\infty e^{-t}t^{z-1}\,dt$. But we can easily see that this integral is absolutely convergent for any z and is an entire function in the finite plane. Thus, for any z we have

$$\Gamma(z) = \sum_0^\infty \frac{(-1)^n}{n!\,(z+n)} + \int_1^\infty e^{-t}t^{z-1}\,dt. \tag{7.49}$$

A transition from (7.47) to (7.49) is actually a substitution of the partial-fraction expansion of $\varphi(z)$ convergent in the entire plane for the integral expression for the meromorphic function $\varphi(z)$. Formula (7.49) is, obviously, the partial-fraction expansion of $\Gamma(z)$.

7.12

THE ORDER AND TYPE OF AN ENTIRE FUNCTION. THEOREMS OF HADAMARD AND BOREL

The most important characteristic of an entire function $f(z)$ is the maximum of its modulus in the circle $|z| \leqslant r$ considered as a function of r:

$$M(r) = \max_{|z| \leqslant r} |f(z)| \quad (0 < r < +\infty).$$

The function $M(r)$ is simply called the *maximum modulus* of the entire function. Since $|f(z)|$ attains its maximal value in the circle $|z| \leqslant r$ at the points on the circle, $|z| = r$ (see Sec. 6.11), we can define $M(r)$ also as the maximum of $|f(z)|$ on the circle:

$$M(r) = \max_{|z|=r} |f(z)|.$$

When $f(z)$ is a polynomial of degree m $(m \geqslant 1)$, i.e.

$$f(z) = a_0 + \ldots + a_m z^m \quad (a_m \neq 0),$$

we have

$$\lim_{|z|\to\infty} \frac{|f(z)|}{|z|^m} = |a_m|,$$

which implies that, for any positive ε,

$$(|a_m| - \varepsilon) r^m < M(r) < (|a_m| + \varepsilon) r^m$$

if r is sufficiently large.

For e^z we find that, on the circle of radius r,

$$|e^z| = \left| \sum_0^\infty \frac{z^n}{n!} \right| \leqslant \sum_0^\infty \frac{r^n}{n!} = e^r;$$

equality is achieved at $z = r$, whence $M(r) = e^r$.

Similarly, for $\sin z$ and $\cos z$ we have

$$|\sin z| = \left| \sum_1^\infty \frac{(-1)^{n-1} z^{2n-1}}{(2n-1)!} \right| \leqslant \sum_1^\infty \frac{r^{2n-1}}{(2n-1)!} = \sinh r,$$

$$|\cos z| \leqslant \cosh r.$$

In both cases equality is achieved at $z = ir$, whence $M(r) = \sinh r$ for $\sin z$ and $M(r) = \cosh r$ for $\cos z$.

From the first definition of the maximum modulus it immediately follows that $M(r)$ is a nondecreasing function, i.e.

$$M(r_1) \leqslant M(r_2) \text{ if } r_1 < r_2.$$

But in view of the maximum modulus principle (see Sec. 6.11) equality is achieved only if $f(z) \equiv \text{const}$. Therefore, for an entire function $f(z)$ that is not a constant, $M(r)$ is a strictly increasing function.

In the general case, if $f(z) \not\equiv \text{const}$, the following proposition is valid:

$$\lim_{r\to\infty} M(r) = \infty,$$

since otherwise we would contradict Liouville's theorem (Sec. 6.2).

If we start from the Cauchy's inequalities used to prove Liouville's theorem (see Sec. 6.2), we arrive at a stronger result: *for every transcendental entire function $f(z)$ (i.e. not a polynomial)*,

$$\overline{\lim_{r\to\infty}} \frac{\ln M(r)}{\ln r} = \infty.$$

Indeed, since otherwise there must exist a positive γ such that

$$M(r) < r^\gamma,$$

for all r sufficiently large. We apply Cauchy's inequalities (6.8) to the expansion coefficients in the power series

$$f(z) = a_0 + a_1 z + \ldots + a_n z^n + \ldots.$$

This yields

$$|a_n| < \frac{r^\gamma}{r^n},$$

whence $a_n = 0$ as $r \to \infty$ only if $n > [\gamma]$. Then $f(z)$ is a polynomial (of a degree no higher than $[\gamma]$), which contradicts our assumption.

Thus, for any transcendental entire function, $\ln M(r)$ is infinitely large in comparison with $\ln r$. This fact justifies the following definition of the *order of an entire function*, i.e. the rate at which $M(r)$ grows: *the order of an entire function $f(z)$ is the number*

$$\rho = \overline{\lim_{r\to\infty}} \frac{\ln \ln M(r)}{\ln r}. \tag{7.50}$$

In view of this definition the order of each of the elementary transcendental entire functions e^z, $\sin z$, $\cos z$, $\sinh z$, and $\cosh z$ proves to be unity. For e^z this is obvious since $M(r)$ here is e^r. Let us calculate the order of $\sin z$. Here

$$M(r) = \sinh r = \frac{e^r - e^{-r}}{2},$$

whence

$$\ln M(r) = r + \ln \frac{1 - e^{-2r}}{2} = r - \ln 2 + \varepsilon_1(r),$$

where $\varepsilon_1(r) \to 0$ as $r \to \infty$. Next,

$$\ln \ln M(r) = \ln r + \ln \left[1 - \frac{\ln 2 - \varepsilon_1(r)}{r}\right] = \ln r + \varepsilon_2(r),$$

where $\varepsilon_2(r) \to 0$ as $r \to \infty$. This implies that the order of $\sin z$ is unity.

The reader will have no difficulty in proving that for $f(z) = e^{z^m}$, where m is a positive integer, the order is m, and for $f(z) = \sin\sqrt{z}/\sqrt{z} = 1 - \frac{z}{3!} + \frac{z^2}{5!} - \dots$ the order is 1/2 (here $M(r) = \sinh\sqrt{r}/\sqrt{r}$). These are examples of functions of finite orders. The simplest example of an entire function of an infinite order is e^{e^z} (here $M(r) = e^{e^r}$).

In general case where $f(z) = \sum\limits_{0}^{\infty} a_n z^n$, the following formula for the order of $f(z)$ can be obtained*:

$$\rho = \overline{\lim_{n\to\infty}} \frac{\ln n}{\ln \dfrac{1}{\sqrt[n]{|a_n|}}}. \tag{7.51}$$

For any number ρ $(0 \leqslant \rho \leqslant +\infty)$ we can select the coefficients a_n in a way such that the sum of the power series is an entire function of order ρ.

The reader is advised to prove that the order ρ of an entire function can be defined as the *greatest lower bound of nonnegative numbers* α

* See, for example, A. I. Markushevich, *The Theory of Analytic Functions* [in Russian], vol. 2, Nauka, Moscow, 1968, p. 250.

for which

$$M(r) < e^{r^\alpha} \text{ at } r > R(\alpha) \tag{7.52}$$

(if this inequality is not valid for all values of α, we assume that $\rho = \infty$).

If the order ρ of a function $f(z)$ is a finite positive number, $0 < \rho < +\infty$, then we can introduce an additional characteristic that specifies more precisely the rate of growth of the maximum modulus. We compare $\ln M(r)$ with r^ρ by introducing the number

$$\sigma = \overline{\lim_{r \to \infty}} \frac{\ln M(r)}{r^\rho}; \tag{7.53}$$

σ is called the *type of an entire function*. If $\sigma = \infty$, we say that the given function of order ρ is a function of the *infinite*, or *maximal*, *type*. If σ is a finite number, then we call $f(z)$ a function of the *finite type*; more precisely, of the *normal*, or *average*, *type* if $0 < \sigma < +\infty$ and of the *minimal type* if $\sigma = 0$.

Each of the functions of the first order e^{Az}, $\sin Az$, and $\cos Az$ $(A \neq 0)$ is of the type $\sigma = |A|$; the function $\exp(a_0 + a_1 z + \ldots + a_m z^m)$ $(a_m \neq 0,\ m \geqslant 1)$ is of order m and type $|a_m|$.

In general, when $f(z) = \sum\limits_0^\infty a_n z^n$ is an entire function of finite positive order ρ, so that (7.51) is valid, the type of this function can be found* from relation

$$(\sigma e \rho)^{\frac{1}{\rho}} = \overline{\lim_{n \to \infty}} n^{\frac{1}{\rho}} \sqrt[n]{|a_n|}. \tag{7.54}$$

The reader can easily prove that the definition of the type of an entire function via formula (7.53) is equivalent to the following: *the type of an entire function of finite positive order* ρ *is the greatest lower bound of positive numbers* β *for which*

$$M(r) < e^{\beta r^\rho} \quad at \quad r > r(\beta).$$

Among the various entire functions those of the *exponential type* have the simplest properties. These are entire functions whose order is either less than or equal to unity and whose type is finite. Among these are $\sin \sqrt{z}/\sqrt{z}$, e^{Az}, $\sin Az$, and $\cos Az$. It is customary to write the power series expansion for a function $f(z)$ of an exponential type as follows:

$$f(z) = \sum_0^\infty \frac{c_n}{n!} z^n \tag{7.55}$$

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* See, for example, A. I. Markushevich, *The Theory of Analytic Functions* [in Russian], vol. 2, Nauka, Moscow, 1968, p. 251.

(by analogy with the power expansion for e^z). Hence, here $a_n = c_n/n!$ and for $\rho = 1$ formula (7.54) yields

$$\sigma e = \overline{\lim_{n\to\infty}} \sqrt[n]{\frac{n^n}{n!}} \sqrt[n]{|c_n|}.$$

But $\lim\limits_{n\to\infty} \sqrt[n]{\dfrac{n^n}{n!}} = e$ (to verify this it suffices to use Stirling's formula: $n! \to \sqrt{2\pi n}\,(n/e)^n$ as $n \to \infty$). This implies that

$$\sigma = \overline{\lim_{n\to\infty}} \sqrt[n]{|c_n|} \quad \text{at } \rho = 1. \tag{7.56}$$

Let us prove the validity of (7.56) independently of (7.54), which was given without proof. Suppose the function represented by the power series (7.55) is an entire function of the exponential type. Then for any positive ε there exists a positive $r(\varepsilon)$ such that

$$M(r) < e^{(\sigma+\varepsilon) r} \quad \text{at} \quad r > r(\varepsilon).$$

(For $\rho < 1$ we assume here that $\sigma = 0$.)

The Cauchy's inequalities for the expansion coefficients in (7.55) are

$$\frac{|c_n|}{n!} \leqslant \frac{M(r)}{r^n} \leqslant \frac{e^{(\sigma+\varepsilon)r}}{r^n} \quad \text{at } r > r(\varepsilon).$$

If we put $r = n/(\sigma+\varepsilon)$, we obtain, for $n > r(\varepsilon)(\sigma+\varepsilon)$,

$$\frac{|c_n|}{n!} \leqslant \left[\frac{e(\sigma+\varepsilon)}{n}\right]^n,$$

whence

$$\overline{\lim_{n\to\infty}} \sqrt[n]{|c_n|} \leqslant e(\sigma+\varepsilon) \lim_{n\to\infty} \sqrt[n]{\frac{n!}{n^n}} = \sigma+\varepsilon.$$

Hence,

$$\overline{\lim_{n\to\infty}} \sqrt[n]{|c_n|} \leqslant \sigma \tag{7.56'}$$

in view of the fact that ε is arbitrarily small. This also implies that $\overline{\lim\limits_{n\to\infty}} \sqrt[n]{|c_n|} = 0$ at $\rho < 1$, since in this case $\sigma = 0$.

Conversely, if we assume that $\overline{\lim\limits_{n\to\infty}} \sqrt[n]{|c_n|} = \sigma' < +\infty$, then for any positive ε we find that $|c_n| < (\sigma'+\varepsilon)^n$ at $n > N(\varepsilon)$ [$N(\varepsilon)$ is a positive integer].

Therefore, for the modulus of the sum of (7.55) we have the following estimate from above:

$$|f(z)| \leqslant \left|\sum_{0}^{N(\varepsilon)}\right| + \left|\sum_{N(\varepsilon)+1}^{\infty}\right| \leqslant \left|\sum_{0}^{N(\varepsilon)} \frac{c_n z^n}{n!}\right| + \sum_{N(\varepsilon)+1}^{\infty} \frac{(\sigma'+\varepsilon)^n |z|^n}{n!}$$

$$< \left|\sum_{0}^{N(\varepsilon)} \frac{c_n z^n}{n!}\right| + e^{(\sigma'+\varepsilon)|z|} < e^{(\sigma'+2\varepsilon)|z|} \text{ if } |z| > R(\varepsilon).$$

Hence

$$M(r) \leqslant e^{(\sigma'+2\varepsilon)r} \text{ if } r > R(\varepsilon).$$

[Because of this the order of $f(z)$ does not exceed 1, and if it is equal to unity, then σ does not exceed $\sigma' + 2\varepsilon$. Due to the fact that ε is arbitrarily small, this implies that

$$\sigma \leqslant \sigma' = \overline{\lim_{n\to\infty}} \sqrt[n]{|c_n|} \quad (\rho = 1). \tag{7.56''}$$

Comparing (7.56′) with (7.56″), we arrive at (7.56).

We can therefore give the following definition of an exponential-type function, a definition equivalent to the one given above:

An analytic function $f(z)$ that can be expanded as in (7.55) *is said to be of the exponential type if*

$$\overline{\lim_{n\to\infty}} \sqrt[n]{|c_n|} < +\infty. \tag{7.57}$$

Returning to the entire functions of an arbitrary finite order, let us note the special features introduced by these functions into the general Weierstrass formula (see Sec. 7.9).

First we introduce the notion of the *exponent of convergence* for any nondecreasing sequence of positive numbers α_n that satisfies the condition $\lim_{n\to\infty} \alpha_n = +\infty$. This is what we call the greatest lower bound τ of positive numbers γ for which $\sum_{1}^{\infty} \frac{1}{\alpha_n^{\gamma}}$ converges. It is easy to see that

$$\tau = \overline{\lim_{n\to\infty}} \frac{\ln n}{\ln \alpha_n}. \tag{7.58}$$

Indeed, suppose $\sum_{1}^{\infty} u_n$ is a converging series with positive nondecreasing terms. Then to each positive δ there corresponds an $N(\delta)$ such that $u_{n+1} + \ldots + u_{n+p} < \delta$ if $n > N(\delta)$, with p a

positive integer, which implies that $pu_{n+p} < \delta$ for $n > N(\delta)$. Assuming that p is n and $n+1$, we have $2nu_{2n} < 2\delta$ and $(2n+1)u_{2n+1} < 2(n+1)u_{2n+1} < 2\delta$ if $n > N(\delta)$. Hence, the condition $\lim\limits_{n\to\infty} nu_n = 0$ is met. For this reason, if the series $\sum\limits_{1}^{\infty} \frac{1}{\alpha_n^{\gamma}}$ converges for a positive γ, then $\lim\limits_{n\to\infty} \frac{n}{\alpha_n^{\gamma}} = 0$, whence $\gamma \ln \alpha_n - \ln n \to +\infty$, and since $\ln \alpha_n \to +\infty$, we find that $\gamma \geqslant \overline{\lim\limits_{n\to\infty}} \frac{\ln n}{\ln \alpha_n} = \tau'$ and $\tau = \inf \gamma \geqslant \tau'$. This implies, for one, that if τ is finite, then τ' is finite, too.

On the other hand, if τ' is finite ($\tau' \geqslant 0$), then for any positive δ we have $\frac{\ln n}{\ln \alpha_n} < \tau' + \delta$ for $n > N(\delta)$, whence $\alpha_n > n^{1/(\tau'+\delta)}$ and $\alpha_n^{\tau'+2\delta} > n^{(\tau'+2\delta)/(\tau'+\delta)}$ for $n > N(\delta)$. But this means that $\sum\limits_{1}^{\infty} \frac{1}{\alpha^{\tau'+2\delta}}$ converges, i.e. $\tau' + 2\delta \geqslant \tau$. Due to the fact that δ is arbitrarily small, we can conclude that $\tau' \geqslant \tau$ or, if we compare this with $\tau \geqslant \tau'$, that $\tau = \tau' = \overline{\lim\limits_{n\to\infty}} \frac{\ln n}{\ln \alpha_n}$.

We will use the notion of the exponent of convergence when speaking of a sequence of complex numbers $\{z_n\}$ that are nonzero and are ordered in a way such that the moduli of the terms are nondecreasing. Namely, we call the exponent of convergence of a sequence consisting of the moduli of terms from the initial sequence the exponent of convergence of the sequence itself.

Hadamard's theorem. *If a function $f(z)$ is of finite order ρ and has an infinitude of zeros, $\underbrace{0, \ldots, 0}_{\lambda}, z_1, \ldots, z_n, \ldots$, where the sequence of zeros is ordered in such a way that their moduli are nondecreasing, then the exponent of convergence τ of these zeros does not exceed ρ.*

The proof of this theorem will be given at the end of Sec. 7.14. For the time being we will use this theorem to note that for any entire function of finite order ρ with an infinitude of zeros there exists a nonnegative integer $\varkappa$ such that the series $\sum\limits_{1}^{\infty} \frac{1}{|z_n|^{\varkappa}}$ diverges while the series $\sum\limits_{1}^{\infty} \frac{1}{|z_n|^{\varkappa+1}}$ converges. It is obvious that $\varkappa + 1 \geqslant \tau \geqslant \varkappa$, whence $\varkappa \leqslant [\tau]$ and, by Hadamard's theorem, $\varkappa \leqslant [\rho]$.

In view of the aforementioned we can assume that all the k_n in the Weierstrass formula (7.27) be equal to $\varkappa$. Then the general formula

takes on the form

$$f(z) = e^{g(z)} z^{\lambda} \prod_{1}^{\infty} \left(1 - \frac{z}{z_n}\right) e^{\frac{z}{z_n} + \dots + \frac{z^{\varkappa}}{\varkappa z_n^{\varkappa}}}.$$

In the case where $\varkappa = 0$, i.e. the series $\sum \frac{1}{|z_n|}$ converges, each of the exponential functions under the product sign is replaced by unity.

The French mathematician J. S. Hadamard proved further that if for a function of finite order ρ the product expansion is taken to be the one just cited, where $\varkappa$ is defined in the above manner, then the entire function $g(z)$ is necessarily a polynomial of degree m not exceeding the integral part of ρ. A theorem that in a certain sense is the converse of Hadamard's results was proved by the French mathematician and politician F. Borel. Namely, if an entire function admits an expansion of the above-mentioned type, where we use $\varkappa$ in the same sense, and if $g(z)$ is a polynomial, then $f(z)$ is an entire function of a finite order. We will combine these two propositions into one theorem, the proof of which we omit.*

Hadamard-Borel's theorem. *If the zeros of an entire function $f(z)$ of finite order ρ form an infinite sequence $\underbrace{0, \dots, 0}_{\lambda}, z_1, \dots, z_n,$ the exponent of convergence τ of this sequence does not exceed ρ, and if $\varkappa$ ($\varkappa \leqslant [\tau] \leqslant [\rho]$) is the largest of the integers k for which $\sum_{1}^{\infty} \frac{1}{|z_n|^k}$ diverges, then*

$$f(z) = z^{\lambda} e^{g(z)} \prod_{1}^{\infty} \left(1 - \frac{z}{z_n}\right) e^{\frac{z}{z_n} + \dots + \frac{z^{\varkappa}}{\varkappa z_n^{\varkappa}}}, \tag{7.59}$$

where $g(z)$ is a polynomial of degree m not higher than $[\rho]$.

Conversely, if (1) the exponent of convergence τ of a sequence of zeros of an entire function $f(z)$ is finite, (2) $\varkappa$ is the greatest integer among the k's for which $\sum \frac{1}{|z_n|^k}$ diverges, and (3) $f(z)$ can be given as a product expansion (7.59), where $g(z)$ is a polynomial of degree m, then $f(z)$ has a finite order $\rho = \max(\tau, m)$.

In the particular case where $\rho < 1$ we must have $\varkappa = 0$ and $m = 0$. The expansion (7.59) takes on the simpler form

$$f(z) = C z^{\lambda} \prod_{1}^{\infty} \left(1 - \frac{z}{z_n}\right),$$

where C is a constant.

* See, for instance, A. I. Markushevich, *The Theory of Analytic Functions* [in Russian], vol. 2, Nauka, Moscow, 1968, pp. 282-7.

[Since the function $f(z) = \sin \sqrt{z}/\sqrt{z}$ is of order 1/2 and the sequence of its zeros is π^2, $4\pi^2$, $9\pi^2$, . . ., we have

$$\frac{\sin \sqrt{z}}{\sqrt{z}} = C \prod_{1}^{\infty} \left(1 - \frac{z}{n^2\pi^2}\right).$$

If we put $z = 0$, we find that $C = 1$. This yields

$$\frac{\sin \sqrt{z}}{\sqrt{z}} = \prod_{1}^{\infty} \left(1 - \frac{z}{n^2\pi^2}\right).$$

By substituting ζ for $\sqrt{z}$, we find the product expansion

$$\sin \zeta = \zeta \prod_{1}^{\infty} \left(1 - \frac{\zeta^2}{n^2\pi^2}\right).$$

7.13

THE PHRAGMÉN-LINDELÖF PRINCIPLE AND FUNCTION

We start by applying the maximum modulus principle (Sec. 6.11) to an entire function. Let us consider $\sin z$ in the upper (or lower) half-plane. The boundary, obviously, is the real axis. Since in such domains $|\sin z| \leqslant 1$ and nowhere at the interior points does $|\sin z|$ attain its maximum, $|\sin z| \leqslant 1$ at all points of each half-plane, whence, by Liouville's theorem, $\sin z$ is a constant. The mistake here lies in the fact that the point at infinity, which belongs to the boundary of each domain, is the point of discontinuity for $\sin z$; sequences of points at which $\sin z$ takes on any values, however large (for instance) in modulus, converge to the point at infinity.

A principle that enables one to arrive at correct results in similar cases was first stated in 1908 by the Swedish analyst L. E. Phragmén and the Finish analyst and topologist E. L. Lindelöf. This principle deals with a function $f(z)$ that is continuous at all finite points of an angular domain G with an opening angle $\alpha\pi$ $(0 < \alpha \leqslant 2)$ and the vertex at z_0 (boundary points included) and is analytic inside the domain (for that matter, it deals with entire functions). The main role is played here by the restriction on the growth of $|f(z)|$ in G as $z \to \infty$. Indeed, it is assumed that there exists a nonnegative $\rho < 1/\alpha$ such that for any $\rho_1 > \rho$ there is an increasing sequence of radii of circles, $\{r_n\}$, $r_n \to +\infty$, all centered at z_0, and that on the arcs of these circles cut off by the sides of the angle the inequality

$$|f(z)| < e^{|z|^{\rho_1}} \tag{7.60}$$

is valid. On these assumptions it is proved that if on the sides of the angle $|f(z)| \leqslant M$, the same is true in the angular domain (equality at interior points is possible if $f(z)$ is a constant).

In the example with which we started this section, $\alpha = 1$ and (7.60) is not valid on any arcs of the semicircles $|z| = r_n$, $r_n \to \infty$, $0 \leqslant \arg z \leqslant \pi$, if ρ_1 is less than unity $\left(\text{this follows from the fact that at points on the imaginary axis } z = ir \text{ and } |f(z)| = \right.$ $\left. = \frac{1}{2}(e^{|z|} - e^{-|z|}) \approx \frac{1}{2}e^{|z|}\right)$.

Note that (7.60) implies that

$$M(r) = \max_{z \in |z - z_0| = r_n \cap \overline{G}} |f(z)|$$

satisfies the condition

$$\frac{\ln \ln M(r_n)}{\ln r_n} \leqslant \rho_1 \text{ as } r_n \to +\infty$$

whence $\varliminf_{r \to +\infty} \frac{\ln \ln M(r)}{\ln r} \leqslant \rho_1$. But ρ_1 is any number greater than ρ, which implies that

$$\varliminf_{r \to +\infty} \frac{\ln \ln M(r)}{\ln r} \leqslant \rho < \frac{1}{\alpha}. \tag{7.61}$$

Obviously, any function of the order $\rho < 1/\alpha$ meets this criterion.

Let us shift the origin of coordinates to the vertex of the angle and take the bisector of this angle as the positive half of the real axis. Then all points of G are characterized thus: $0 < |z| < +\infty$, $|\arg z| < \alpha\pi/2$.

Next we apply the mapping $\zeta = z^{1/\alpha}$, which transforms the domain into the right half-plane: $|\arg \zeta| < \pi/2$ and $f(z)$ into $\varphi(\zeta) = f(\zeta^\alpha)$. Then $\mu(r) = \max\limits_{|\zeta| = r,\ |\arg \zeta| \leqslant \frac{\pi}{2}} |\varphi(\zeta)| = M(r^\alpha)$ and (7.61) becomes

$$\varliminf_{r \to +\infty} \frac{\ln \ln \mu(r)}{\ln r} = \varliminf_{r \to \infty} \alpha \frac{\ln \ln M(r^\alpha)}{\ln r^\alpha} \leqslant \alpha\rho < 1 \tag{7.62}$$

(if $\mu(r) < 1$ so that $\ln \mu(r) < 0$, then we must substitute 1 for such values of $\mu(r)$).

Now we can formulate and prove the Phragmén-Lindelöf principle without restricting its applicability.

The Phragmén-Lindelöf principle. *We assume that $\varphi(z)$ is a function that is continuous at all finite points of the right half-plane G, $0 \leqslant |z| < +\infty$ (the boundary points included), analytic inside it,*

and satisfies the condition

$$\varlimsup_{r\to\infty} \frac{\ln \ln \mu(r)}{\ln r} = \lambda < 1, \qquad (7.62')$$

where $\mu(r) = \max\limits_{|z|=r,\ |\arg z|\leqslant \frac{\pi}{2}} |\varphi(z)|.$

Then from the fact that $|\varphi(z)| \leqslant M$ *at all finite boundary points of* G *it follows that the same is true for all interior points of* G; *equality is possible only if* $\varphi(z)$ *is a constant.*

Proof. Suppose that $\lambda' < \lambda''$ are two numbers that lie between λ and 1: $0 \leqslant \lambda < \lambda' < \lambda'' < 1$. In view of (7.62′) there exists an increasing sequence $\{r_n\}$, $r_n \to \infty$, such that

$$\frac{\ln \ln \mu(r_n)}{\ln r_n} < \lambda'. \qquad (7.62'')$$

For an arbitrarily small positive number ε we build an auxiliary function

$$F(z) = e^{-\varepsilon z^{\lambda''}} \varphi(z).$$

Assuming that $z = re^{i\theta}$, $|\theta| \leqslant \pi/2$, we have

$$|F(re^{i\theta})| = e^{-\varepsilon r^{\lambda''} \cos \lambda''\theta} |\varphi(re^{i\theta})| \leqslant |\varphi(re^{i\theta})|$$

because $0 < \lambda'' < 1$ and $|\theta| \leqslant \pi/2$. Therefore, on the boundary of G the function $F(z)$, just as $\varphi(z)$, satisfies the condition

$$|F(z)| \leqslant M.$$

We fix $z_0 \in G$ and consider the semicircle K_n: $|z| \leqslant r_n$, $|\arg z| \leqslant \leqslant \pi/2$, where $r_n > r_0 = |z_0|$. On the semicircle, in view of (7.62″),

$$|\varphi(r_n e^{i\theta})| \leqslant \mu(r_n) < e^{r_n^{\lambda'}},$$

whence, assuming for brevity that $\cos(\pi\lambda''/2) = \delta > 0$, we find that

$$|F(r_n e^{i\theta})| = e^{-\varepsilon r_n^{\lambda''} \cos \lambda''\theta} |\varphi(r_n e^{i\theta})| < \exp(-\varepsilon\delta r_n^{\lambda''} + r_n^{\lambda'}) \to 0$$

$$\text{as } r_n \to \infty.$$

Hence, for sufficiently great n's,

$$|F(r_n e^{i\theta})| \leqslant M, \quad |\theta| \leqslant \frac{1}{2}\pi.$$

Thus, we can apply the maximum modulus principle (more precisely, a corollary of this principle stated on p. 205) to the function $F(z)$

in a semicircle K_n. This implies that at point z_0 ($z_0 \in K_n$),

$$|F(z_0)| = e^{-\varepsilon r_0^{\lambda''} \cos \lambda''\theta_0} |\varphi(r_0 e^{i\theta_0})| \leqslant M.$$

As $\varepsilon \to 0$, this becomes

$$|\varphi(r_0 e^{i\theta_0})| \leqslant M.$$

The equality sign means that $|\varphi(z)|$ attains a maximum at an interior point of G, which implies that $\varphi(z)$ is a constant in this case. The proof is complete.

Note that due to the above remarks the Phragmén-Lindelöf principle can be applied to any entire function $f(z)$ of finite order ρ in any angular domain with the vertex at the origin of coordinates provided that the opening angle $\alpha\pi$ is such that $\rho < 1/\alpha$, i.e. $0 < \alpha < 1/\rho$. Suppose, for instance, that $\rho < 1/2$. Then we can put $\alpha = 2$. The angular domain G is the complex plane with a deleted ray starting at point $z = 0$. If the function is bounded on the ray, then by the above principle it is bounded in the entire plane, i.e. it is a constant. Thus, *if an entire function $f(z)$ is not a constant and its order is less than 1/2, it is unbounded on any ray starting at point $z = 0$.*

This statement is not true, however, for a function of order 1/2, as the example of $\sin\sqrt{z}/\sqrt{z}$ shows (this function tends to zero as $z \to \infty$ along the positive half of the real axis).

Now let the order of $f(z)$ be greater than 1/2 (but less than $+\infty$). From the origin of coordinates we draw a system of rays in a way such that the angle between two adjacent rays is smaller than π/ρ. If on each ray the function is bounded, then we will find that it is a constant, just as in the previous case. But this is impossible because its order is positive. Hence, $f(z)$ is unbounded at least on one of the rays of the system. For instance, if $\rho < 1$, then for a function of order ρ not to be a constant it must be unbounded on at least one of the two rays into which point $z = 0$ divides any straight line passing through it.

For a more detailed examination of the behavior of an entire function, another quantity (aside from $M(r)$) is used to characterize the growth of the function along each of the rays starting at the origin of coordinates. Suppose that $f(z)$ is an entire function of finite order $\rho > 0$ and of finite type σ. Then to each ray that is at an angle θ to the positive half of the real axis there corresponds a quantity

$$h(\theta) = \overline{\lim_{r\to\infty}} \frac{\ln |f(re^{i\theta})|}{r^\rho}. \tag{7.63}$$

This is a single-valued and periodic function with a period of 2π; it is known as the *Phragmén-Lindelöf function.*

Since

$$\overline{\lim_{r\to\infty}} \frac{\ln M(r)}{r^\rho} = \sigma < \infty,$$

we have

$$h(\theta) \leqslant \sigma,$$

i.e. $h(\theta)$ is bounded from above.

As an example let us calculate $h(\theta)$ for $f(z) = e^z$. Here $\rho = 1$ and $\sigma = 1$; whence

$$h(\theta) = \overline{\lim_{r\to\infty}} \frac{\ln |f(re^{i\theta})|}{r} = \lim_{r\to\infty} \frac{r\cos\theta}{r} = \cos\theta.$$

It is positive in the right half-plane and negative in the left, since $f(z)$ tends to ∞ along rays in the right half-plane and to 0 in the left.

The Phragmén-Lindelöf principle enables us to derive the basic property of $h(\theta)$, which is expressed by the following

Theorem. *If θ_1 and θ_2 are such that*

$$0 < \theta_2 - \theta_1 < \min\left(\frac{\pi}{\rho}, 2\pi\right) \tag{7.64}$$

and if a and b are finite numbers such that $h(\theta_1) \leqslant a$ and $h(\theta_2) \leqslant b$, then in the interval (θ_1, θ_2) we find that

$$h(\theta) \leqslant A\cos\theta\rho + B\sin\theta\rho, \tag{7.65}$$

where A and B are determined via the set of equations

$$\begin{aligned} A\cos\theta_1\rho + B\sin\theta_1\rho &= a,\\ A\cos\theta_2\rho + B\sin\theta_2\rho &= b. \end{aligned} \tag{7.66}$$

Moreover, for any positive ε we have

$$\ln|f(re^{i\theta})| < (A\cos\theta\rho + B\sin\theta\rho + \varepsilon)\, r^\rho \tag{7.67}$$

if $\theta_1 \leqslant \theta \leqslant \theta_2$ and $r > R(\varepsilon)$.

Proof. Suppose that η is an arbitrary positive number. We find $\alpha = \alpha(\eta)$ and $\beta = \beta(\eta)$ from

$$\alpha\cos\theta_1\rho + \beta\sin\theta_1\rho = a + \eta,$$
$$\alpha\cos\theta_2\rho + \beta\sin\theta_2\rho = b + \eta$$

(this is possible because the system determinant $\sin(\theta_2 - \theta_1)\rho$ is nonzero due to the restrictions imposed on θ_1 and θ_2). Obviously, $\lim\limits_{\eta\to 0} \alpha(\eta) = A$ and $\lim\limits_{\eta\to 0} \beta(\eta) = B$.

Let us introduce the auxiliary function

$$F_\eta(z) = e^{-(\alpha - i\beta)z^\rho} f(z),$$

which is single-valued (if we put $z^\rho = r^\rho(\cos\theta\rho + i\sin\theta\rho)$), analytic in the angular domain g: $0 < r < \infty$, $\theta_1 < \theta < \theta_2$, and

in this domain satisfies the inequality

$$| F_\eta (re^{i\theta}) | = e^{-(\alpha \cos \theta\rho + \beta \sin \theta\rho) r^\rho} | f(re^{i\theta}) |$$
$$< \exp [(\eta + h(\theta)) r^\rho - (\alpha \cos \theta\rho + \beta \sin \theta\rho) r^\rho] < \exp (K r^\rho)$$

at $r > R_1(\eta)$. Besides, due to the definition of a and b and the choice of α and β we find that, on the sides of the angle,

$$| F_\eta (re^{i\theta_1}) | \leqslant \exp [(a + \eta) r^\rho - (\alpha \cos \theta_1\rho + \beta \sin \theta_1\rho) r^\rho] = 1,$$
$$| F_\eta (re^{i\theta_2}) | \leqslant \exp [(b + \eta) r^\rho - (\alpha \cos \theta_2\rho + \beta \sin \theta_2\rho) r^\rho] = 1$$

at $r > R_1(\eta)$. For this reason $| F_\eta (z) |$ is bounded on the sides of angle g:

$$| F_\eta (re^{i\theta_1}) | < M, \quad | F_\eta (re^{i\theta_2}) | < M, \quad 0 < r < +\infty,$$

where $M = M_\eta$.

Since due to (7.64) the order ρ and the opening angle of g, $\theta_2 - \theta_1 = \pi \frac{\theta_2 - \theta_1}{\pi}$, are related thus:

$$\theta_2 - \theta_1 < \frac{\pi}{\rho}, \quad \text{or} \quad \rho < \frac{\pi}{\theta_2 - \theta_1},$$

to $F_\eta (re^{i\theta})$ we can apply the Phragmén-Lindelöf principle. We have

$$| F_\eta (z) | \leqslant M_\eta, \quad z \in \bar{g},$$

whence

$$\ln | f(re^{i\theta}) | \leqslant \ln M_\eta + (\alpha \cos \theta\rho + \beta \sin \theta\rho) r^\rho, \quad re^{i\theta} \in \bar{g}.$$

For a given positive ε we fix the number $\eta = \eta(\varepsilon) > 0$ in a way such that

$$| \alpha(\eta) - A | < \frac{\varepsilon}{3} \quad \text{and} \quad | \beta(\eta) - B | < \frac{\varepsilon}{3},$$

and then find an $R(\varepsilon)$ such that for $r > R(\varepsilon)$ the inequality

$$\frac{\ln | M_\eta |}{r^\rho} < \frac{\varepsilon}{3}$$

is valid. According to the aforesaid, for $\theta_1 < \theta < \theta_2$ and $r > R(\varepsilon)$,

$$\ln | f(re^{i\theta}) | < (A \cos \theta\rho + B \sin \theta\rho + \varepsilon) r^\rho,$$

which is the sought for relationship. It also implies that

$$h(\theta) = \overline{\lim_{r \to \infty}} \frac{\ln | f(re^{i\theta}) |}{r^\rho} \leqslant A \cos \theta\rho + B \sin \theta\rho + \varepsilon,$$

or, since ε is arbitrarily small,

$$h(\theta) \leqslant A \cos \theta\rho + B \sin \theta\rho \quad (\theta_1 \leqslant \theta \leqslant \theta_2).$$

The proof is complete.

We find A and B via Eqs. (7.66) and substitute them into (7.65); this yields

$$h(\theta)=\frac{a\sin[(\theta_2-\theta)\rho]+b\sin[(\theta-\theta_1)\rho]}{\sin[(\theta_2-\theta_1)\rho]}\quad(\theta_1\leqslant\theta\leqslant\theta_2).\quad(7.68)$$

In particular, on the bisector of g we have

$$h\left(\frac{\theta_1+\theta_2}{2}\right)\leqslant\frac{a+b}{2\cos\frac{\theta_2-\theta_1}{2}\rho}.\quad(7.68')$$

Let us show that *the Phragmén-Lindelöf function is finite for any value of* θ:

$$h(\theta)>-\infty.\quad(7.69)$$

Suppose that the opposite is true, and put $h(\theta_0)=-\infty$. We fix a positive integer $m>\rho$ and supply (7.68′) to the bisector of the angle $\theta_0<\theta<\theta_0+\pi/m$. We have

$$h\left(\theta_0+\frac{\pi}{2m}\right)\leqslant\frac{a+b}{2\cos\frac{\pi\rho}{2m}},$$

where a and b are any two finite numbers for which $a\geqslant h(\theta_0)=-\infty$ and $b\geqslant h(\theta_1)$. We fix b and go over to the limit as $a\to-\infty$. This yields $h\left(\theta_0+\frac{\pi}{2m}\right)=-\infty$. If we repeat this line of reasoning, we find that in general $h\left(\theta_0+k\frac{\pi}{2m}\right)=-\infty$, $k=0, 1, \ldots, 4m-1$. We obtain a system of rays on each of which $f(z)$ is bounded (due to the definition of $h(\theta)$). Since the angle between two adjacent rays, which is π/m, is less than π/ρ, by the aforesaid $f(z)$ is a constant. But this contradicts the initial assumption that $f(z)$ is an entire function of order $\rho>0$.

It follows from the proved relationship that in (7.68) we can simply put $a=h(\theta_1)$ and $b=h(\theta_2)$. For this reason $h(\theta)$ always obeys the following inequality:

$$h(\theta)\leqslant\frac{h(\theta_1)\sin[(\theta_2-\theta)\rho]+h(\theta_2)\sin[(\theta-\theta_1)\rho]}{\sin[(\theta_2-\theta_1)\rho]},\quad(7.70)$$

where $\theta_1\leqslant\theta\leqslant\theta_2$ and $0<\theta_2-\theta_1<\min(\pi/\rho, 2\pi)$.

In the particular case of $\rho=1$ this inequality expresses the *trigonometric convexity* of the corresponding function $h(\theta)$.

From (7.70) it follows that *the Phragmén-Lindelöf function of an entire function of a finite order is continuous in* θ.

Indeed, let us fix an arbitrary θ_0, put $\theta_1=\theta_0$ in (7.70), and go over to the limit as $\theta\to\theta_0$. We have

$$\varlimsup_{\theta\to\theta_0,\ \theta>\theta_0}h(\theta)\leqslant h(\theta_0).$$

If we put $\theta_2 = \theta_0$ and again go over to the limit as $\theta \to \theta_0$, we have

$$\overline{\lim_{\theta\to\theta_0,\ \theta<\theta_0}} h(\theta) \leqslant h(\theta_0).$$

Hence

$$\overline{\lim_{\theta\to\theta_0}} h(\theta) \leqslant h(\theta_0).$$

Again we turn to (7.70), put $\theta = \theta_0$, and go over to the limit, first

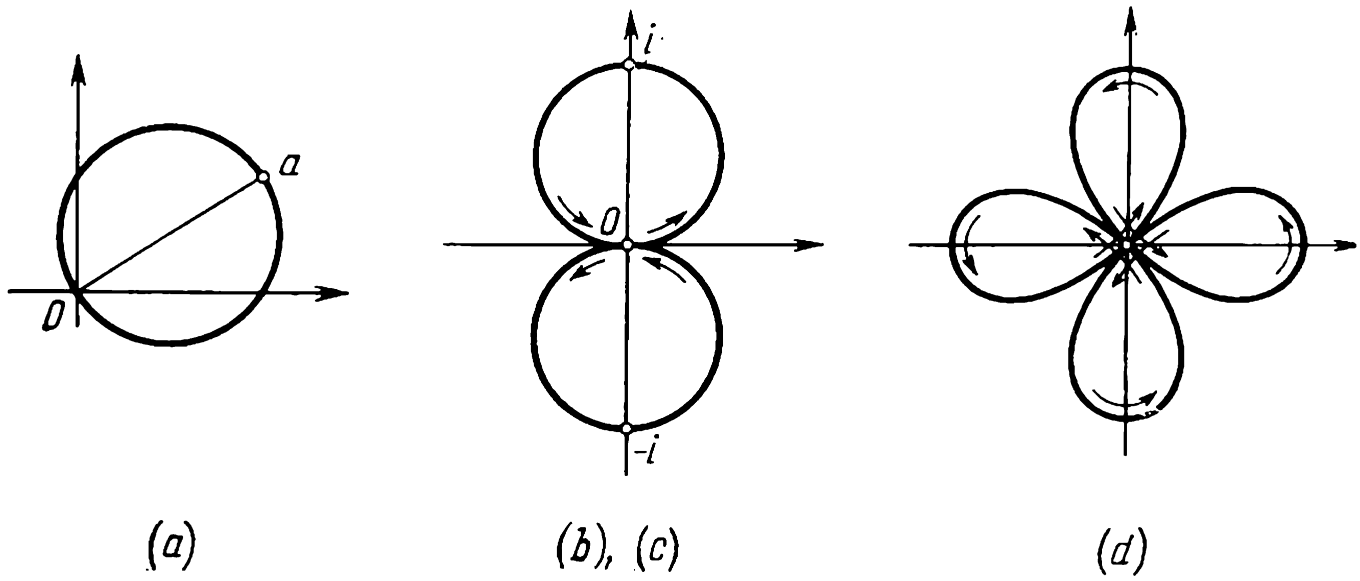


Fig. 53

as $\theta_1 \to \theta_0$ and then as $\theta_2 \to \theta_0$. We have

$$h(\theta_0) \leqslant \underline{\lim_{\theta_1\to\theta_0,\ \theta_1<\theta_0}} h(\theta_1), \quad h(\theta_0) \leqslant \underline{\lim_{\theta_2\to\theta_0,\ \theta_2>\theta_0}} h(\theta_2),$$

i.e.

$$\underline{\lim_{\theta\to\theta_0}} h(\theta) \geqslant h(\theta_0).$$

Comparing the above inequalities, we find that

$$\overline{\lim_{\theta\to\theta_0}} h(\theta) = \underline{\lim_{\theta\to\theta_0}} h(\theta) = h(\theta_0),$$

which is what we set out to prove.

In polar coordinates r and θ, the $r = h(\theta)$ versus θ diagram of the Phragmén-Lindelöf function is sometimes called the *indicatrix* of the particular entire function.

Curves (*a*), (*b*), (*c*), and (*d*) in Fig. 53 depict the indicatrices for e^{az}, $\sin z$, $\cos z$, and e^{z^2}.

7.14

JENSEN'S FORMULA

Let $f(z)$ be a meromorphic function that is not a constant. We order its zeros in a way such that their moduli are nondecreasing, and we do the same to its nonzero poles. As a result we have two sequences: a sequence $\{a_j\}$ of the zeros, with $0 < |a_j| \leqslant |a_{j+1}|$, $j = 1, 2, \ldots$, and a sequence $\{b_k\}$ of the nonzero poles, with $0 < |b_k| \leqslant |b_{k+1}|$, $k = 1, 2, \ldots$; each point is repeated in a sequence as many times as the order of the particular zero or pole. In a neighborhood of the origin of coordinates the Laurent expansion for $f(z)$ is $f(z) = \sum\limits_{j=\lambda}^{\infty} c_j z^j$, $c_\lambda \neq 0$; for $\lambda > 0$ the point $z = 0$ is a zero of order λ, and for $\lambda < 0$ a pole of order $-\lambda$.

We will now establish an important relationship between the zeros and poles of the function that lie in a circle $K\{|z| < R\}$ of an arbitrary radius, on the one hand, and the values of the modulus of the function on the circle $|z| = R$, on the other. To simplify matters we assume that $f(z)$ has neither zeros nor poles on the circle $|z| = R$.

Suppose points $a_1, \ldots, a_n$ and $b_1, \ldots, b_p$ are the zeros and poles that lie in K. We take one of these points, say b_k, and build a rational function with no poles in K, that has only one simple zero at b_k, and whose modulus is identically 1 on the boundary of K. The simplest function is the one that conformally maps K into the unit circle, with point b_k being mapped, in the process, into the origin of coordinates:

$$w = R\frac{z - b_k}{R^2 - \bar{b}_k z}$$

(see formula (3.15)). Indeed, this function has a single zero at b_k, its single pole $R^2/\bar{b}_k$ lies in the exterior of K, and, finally, at $|z| = R$, i.e. at $z\bar{z} = R^2$,

$$\left| R\frac{z - b_k}{R^2 - \bar{b}_k z} \right| = \frac{R}{|z|}\left| \frac{z - b_k}{\frac{R^2}{z} - \bar{b}_k} \right| = \frac{R}{|z|}\left| \frac{z - b_k}{\bar{z} - \bar{b}_k} \right| = 1.$$

It is clear now that the rational function

$$\varphi(z) = \frac{z^\lambda}{R^\lambda} \frac{\prod\limits_1^n R\frac{z - a_j}{R^2 - \bar{a}_j z}}{\prod\limits_1^p R\frac{z - b_k}{R^2 - \bar{b}_k z}}$$

has the same zeros and poles in K as $f(z)$, and $|\varphi(z)| = 1$ for $|z| = R$.

For this reason the function $f(z)/\varphi(z)$ has neither zeros nor poles in K, and its modulus coincides with $|f(z)|$ on the boundary of K. Noting that $\ln \frac{f(z)}{\varphi(z)}$ is analytic in K and, hence, its real part is harmonic, we find that

$$\ln \left| \frac{f(z)}{\varphi(z)} \right|_{z=0} = \ln \left| \frac{f(z)}{z^{\lambda}} R^{\lambda} \right|_{z=0} + \ln \prod_{1}^{p} \frac{|b_k|}{R} - \ln \prod_{1}^{n} \frac{|a_j|}{R}$$
$$= \frac{1}{2\pi} \int_{0}^{2\pi} \ln |f(Re^{i\alpha})| \, d\alpha,$$

by the property of the arithmetic mean (see p. 173), or

$$\ln \prod_{1}^{n} \frac{R}{|a_j|} - \ln \prod_{1}^{p} \frac{R}{|b_k|} + \lambda \ln R$$
$$= \frac{1}{2\pi} \int_{0}^{2\pi} \ln \left| \frac{f(Re^{i\alpha})}{c_{\lambda}} \right| d\alpha. \quad (7.71)$$

This is the sought for relation, known as *Jensen's formula* (named after J.L.W.V. Jensen, a Danish analyst, algebraist, and engineer, who discovered this formula in 1899). Let us find the same formula in a form that is more suitable for application. To this end we introduce the function

$$n(t, f) = n(t, \infty),$$

which gives the number of poles of $f(z)$ (with due regard to the order of each pole) in the circle $|z| \leqslant t$; here $n(0, \infty)$ stands for the order of a possible pole at point $z = 0$. Then

$$n\left(t, \frac{1}{f}\right) = n(t, 0)$$

is the number of poles of $1/f(z)$, i.e. the number of zeros of $f(z)$ in the same circle $|z| \leqslant t$, with $n(0, 0)$ the order of a possible zero at point $z = 0$.

If we write $\prod\limits_{1}^{p} \frac{R}{|b_k|}$ as

$$\frac{|b_2|}{|b_1|} \frac{|b_3|^2}{|b_2|^2} \cdots \frac{|b_p|^{p-1}}{|b_{p-1}|^{p-1}} \frac{R^p}{|b_p|^p},$$

we find that

$$\ln \prod_{1}^{p} \frac{R}{|b_k|} = \sum_{k=1}^{p-1} k \ln \frac{|b_{k+1}|}{|b_k|} + p \ln \frac{R}{|b_p|}$$

$$= \sum_{k=1}^{p-1} \int_{|b_k|}^{|b_{k+1}|} \frac{k\,dt}{t} + \int_{|b_p|}^{R} \frac{p\,dt}{t}.$$

To write the last sum in the form of an integral from 0 to R, we note that only terms for which $|b_k| < |b_{k+1}|$ are nonzero in this sum. But $k = n(t, \infty) - n(0, \infty)$ in such an interval (the origin of coordinates is not included in the poles $b_1, b_2, \ldots, b_k$). Moreover, $n(t, \infty) = n(0, p)$ if $0 < t < |b_1|$ and $p = n(t, \infty) - n(0, \infty)$ if $|b_p| < t \leqslant R$. Whence

$$\ln \prod_{1}^{p} \frac{R}{|b_k|} = \int_{0}^{R} \frac{n(t, \infty) - n(0, \infty)}{t}\,dt.$$

Similarly,

$$\ln \prod_{1}^{n} \frac{R}{|a_j|} = \int_{0}^{R} \frac{n(t, 0) - n(0, 0)}{t}\,dt.$$

Therefore, Jensen's formula takes on the following form:

$$\int_{0}^{R} \frac{n(t, 0) - n(0, 0)}{t}\,dt - \int_{0}^{R} \frac{n(t, \infty) - n(0, \infty)}{t}\,dt + \lambda \ln R = \frac{1}{2\pi} \int_{0}^{2\pi} \ln \left| \frac{f(Re^{i\alpha})}{c_\lambda} \right| d\alpha. \quad (7.72)$$

In the particular case where $f(z)$ is an entire function, $n(t, \infty) \equiv \equiv 0$ and we arrive at a simpler version of this formula:

$$\int_{0}^{R} \frac{n(t, 0) - n(0, 0)}{t}\,dt + \lambda \ln R = \frac{1}{2\pi} \int_{0}^{2\pi} \ln \left| \frac{f(Re^{i\alpha})}{c_\lambda} \right| d\alpha. \quad (7.72')$$

If we put $\max |f(Re^{i\alpha})| = M(R)$, $0 \leqslant \alpha \leqslant 2\pi$, we get *Jensen's inequality*:

$$\int_{0}^{R} \frac{n(t, 0) - n(0, 0)}{t}\,dt \leqslant \ln \frac{M(R)}{|c_\lambda| R^\lambda}. \quad (7.73)$$

This shows that the number of zeros of an entire function grows rapidly in an infinitely expanding circle only if the maximum modu-

lus of the function grows rapidly. To get a clear picture of this interdependence we fix a positive θ, $0 < \theta < 1$. Then

$$\int_0^R \frac{n(t, 0)-n(0, 0)}{t}\, dt \geqslant \int_{\theta R}^R \frac{n(t, 0)-n(0, 0)}{t}\, dt \geqslant [n(\theta R)-n(0, 0)] \ln\frac{1}{\theta}$$

and (7.73) yields

$$n(\theta R, 0)-n(0, 0) \leqslant \frac{\ln\dfrac{M(R)}{|c_\lambda| R^\lambda}}{\ln\dfrac{1}{\theta}}. \qquad (7.73')$$

This is a direct estimate from above of the number of zeros of an entire function in a circle of radius θR in terms of the maximum modulus in a circle of radius R. If $f(z)$ has a finite order ρ, then

$$M(R) < e^{r^{\rho+\varepsilon}}, \quad R > R_0(\varepsilon),$$

for any positive ε (formula (7.52)). If in (7.73′) we put $\theta R = t$, substitute $(t/\theta)^{\rho+\varepsilon}$ for $\ln M(R)$, divide both sides of the inequality by $t^{\rho+\varepsilon}$, and tend t to ∞, we find that

$$\overline{\lim_{t\to\infty}} \frac{n(t, 0)}{t^{\rho+\varepsilon}} \leqslant \frac{1}{\theta^{\rho+\varepsilon} \ln\dfrac{1}{\theta}}.$$

Here the right-hand side has a minimum in the interval (0, 1) at $\theta = e^{-1/(\rho+\varepsilon)}$. Whence

$$\overline{\lim_{t\to\infty}} \frac{n(t, 0)}{t^{\rho+\varepsilon}} \leqslant e(\rho+\varepsilon).$$

Hence, for an entire function of order ρ, the number of zeros in the circle $|z| < t$ asymptotically does not exceed $e(\rho+\varepsilon)\, t^{\rho+\varepsilon}$ $(\varepsilon > 0)$.

From here we can easily obtain Hadamard's theorem, by which we can estimate the exponent of convergence τ of the sequence of zeros of an entire function of finite order ρ:

$$\tau \leqslant \rho$$

(see Sec. 7.12). Indeed, since $n(|a_\nu|, 0)$ is the number of zeros of $f(z)$ that are not at the origin of coordinates and lie in the closed circle $|z| \leqslant |a_\nu|$, we find that $\nu \leqslant n(|a_\nu|, 0)$ (aside from a_ν, there may be zeros with greater subscripts lying on the circle $|z| = |a_\nu|$). Therefore, if we go from t to $|a_\nu|$, the last inequality yields

$$\overline{\lim_{t\to\infty}} \frac{\nu}{|a_\nu|^{\rho+\varepsilon}} \leqslant e(\rho+\varepsilon),$$

or

$$\frac{\nu}{|a_\nu|^{\rho+\varepsilon}} < e(\rho+2\varepsilon) \quad \text{for} \quad \nu > N(\varepsilon).$$

Whence, $|a_\nu|^{\rho+2\varepsilon} > C(\varepsilon)\,\nu^{(\rho+2\varepsilon)/(\rho+\varepsilon)}$ for $\nu > N(\varepsilon)$, i.e. $\sum\limits_1^\infty \frac{1}{|a_\nu|^{\rho+2\varepsilon}}$ converges for any positive ε. Therefore, $\tau \leqslant \rho+2\varepsilon$ and, finally, $\tau \leqslant \rho$, which is what Hadamard proved.

7.15
NEVANLINNA'S FIRST FUNDAMENTAL THEOREM

Let us return to the general case of a meromorphic function. We note that the exponent in the lowest term of the Laurent expansion or this function can be written as

$$\lambda = n(0, 0) - n(0, \infty).$$

Indeed, if $\lambda > 0$, then $n(0, \infty) = 0$ and $\lambda = n(0, 0)$; if $\lambda < 0$, then $n(0, 0) = 0$ and $\lambda = -n(0, \infty)$; finally, if $\lambda = 0$, then $n(0, 0) = n(0, \infty) = 0$.

Let us introduce another function:

$$\ln^+ t = \begin{cases} \ln t & \text{if } t \geqslant 1, \\ 0 & \text{if } 0 < t < 1. \end{cases}$$

It is obvious that $\ln t = \ln^+ t - \ln^+ (1/t)$ for all positive t's. We also note that

$$\ln^+ (t + \tau) \leqslant \ln^+ (2\tau) \leqslant \ln 2 + \ln^+ \tau \leqslant \ln 2 + \ln^+ t + \ln^+ \tau$$

for all values of t and τ $(0 < t \leqslant \tau)$. We can now write Jensen's formula (7.72) thus:

$$\int_0^R \frac{n(t, 0) - n(0, 0)}{t}\,dt - \int_0^R \frac{n(t, \infty) - n(0, \infty)}{t}\,dt + [n(0, 0)$$
$$-n(0, \infty)] \ln R = \frac{1}{2\pi} \int_0^{2\pi} \Big[\ln^+ |f(Re^{i\alpha})|$$
$$- \ln^+ \left|\frac{1}{f(Re^{i\alpha})}\right| \Big] d\alpha - \ln |c_\lambda|,$$

or

$$\frac{1}{2\pi}\int_0^{2\pi} \ln^+ | f(Re^{i\alpha}) | \,d\alpha + \int_0^R \frac{n(t, \infty) - n(0, \infty)}{t}\,dt$$

$$+ n(0, \infty)\ln R - \ln | c_\lambda | = \frac{1}{2\pi}\int_0^{2\pi} \ln^+ \frac{1}{| f(Re^{i\alpha}) |}\,d\alpha$$

$$+ \int_0^R \frac{n(t, 0) - n(0, 0)}{t}\,dt + n(0, 0)\ln R. \quad (7.74)$$

In this form Jensen's formula exhibits a remarkable symmetry between values of the meromorphic functions that are great in modulus (close to ∞ or even equal to ∞) and those that are small (close to zero or even equal to zero). This regularity can be given in a more general form if in (7.74) we substitute $f(z) - a$ for $f(z)$, where a is any finite complex number. We note that these two functions have the same poles, so that $n(t, f - a) = n(t, f) = n(t, \infty)$, and that the zero points of $f(z) - a$ are the a-points of $f(z)$, so that $n\left(t, \frac{1}{f-a}\right) = n(t, a)$ is the number of a-points of $f(z)$ in the circle of radius t. Finally, suppose $c(a)$ is the coefficient in the lowest term of the Laurent expansion for $f(z) - a$ at $z = 0$. Then (7.74) yields

$$\frac{1}{2\pi}\int_0^{2\pi} \ln^+ | f(Re^{i\alpha}) - a | \,d\alpha + \int_0^R \frac{n(t, \infty) - n(0, \infty)}{t}\,dt$$

$$+ n(0, \infty)\ln R - \ln | c(a) | = \frac{1}{2\pi}\int_0^{2\pi} \ln^+ \frac{1}{| f(Re^{i\alpha}) - a |}\,d\alpha$$

$$+ \int_0^R \frac{n(t, a) - n(0, a)}{t}\,dt + n(0, a)\ln R. \quad (7.74')$$

But

$$\ln^+ | f | = \ln^+ | a + (f - a) | \leqslant \ln 2 + \ln^+ | a | + \ln^+ | f - a |$$

and

$$\ln^+ | f - a | \leqslant \ln^+ (| f | + | a |) \leqslant \ln 2 + \ln^+ | f | + \ln^+ | a |,$$

whence

$$\ln^+ | f - a | = \ln^+ | f | + \theta_1 (\ln 2 + \ln | a |),$$

where $-1 \leqslant \theta_1 \leqslant 1$. Therefore, from (7.74′) we have

$$\frac{1}{2\pi}\int_0^{2\pi} \ln^+ \frac{1}{|f(Re^{i\alpha})-a|}\,d\alpha + \int_0^R \frac{n(t,\ a)-n(0,\ a)}{t}\,dt$$

$$+n(0,\ a)\ln R = \frac{1}{2\pi}\int_0^{2\pi} \ln^+ |f(Re^{i\alpha})|\,d\alpha$$

$$+\int_0^R \frac{n(t,\ \infty)-n(0,\ \infty)}{t}\,dt + n(0,\ \infty)\ln R + O(1), \qquad (7.75)$$

where

$$O(1) = \theta(\ln 2 + \ln^+ |a|) - \ln |c(a)|, \quad -1 \leqslant \theta \leqslant 1,$$

is a function of a and R bounded as $R \to \infty$. If we introduce the notation

$$\frac{1}{2\pi}\int_0^{2\pi} \ln^+ |f(Re^{i\alpha})|\,d\alpha + \int_0^R \frac{n(t,\ \infty)-n(0,\ \infty)}{t}\,dt + n(0,\ \infty)\ln R = T(R),$$

we arrive at

Nevanlinna's first fundamental theorem (1924). *For any complex number a, the sum*

$$\frac{1}{2\pi}\int_0^{2\pi} \ln^+ \frac{1}{|f(Re^{i\alpha})-a|}\,d\alpha + \int_0^R \frac{n(t,\ a)-n(0,\ a)}{t}\,dt + n(0,\ a)\ln R$$

remains approximately constant and equal to $T(R)$.

The above sum characterizes the degree to which the values of the meromorphic function $f(z)$ inside the circle $|z| \leqslant R$ are close to a. It takes into account the approximation of $f(z)$ by a in the mean on the circle $|z| = R$ (by means of an integral) and the number of a-points in the circle $|z| \leqslant R$. The above theorem is correct to within a bounded term.

The function $T(R)$ is the *order function* (or *characteristic function*) *of the meromorphic function $f(z)$*. It consists of two terms,

$$m(R,\ f) = m(R,\ \infty) = \frac{1}{2\pi}\int_0^{2\pi} \ln^+ |f(Re^{i\alpha})|\,d\alpha$$

and

$$N(R,\ f) = N(R,\ \infty) = \int_0^R \frac{n(t,\ \infty)-n(0,\ \infty)}{t}\,dt + n(0,\ \infty)\ln R.$$

By using these notations (all were introduced by R. Nevanlinna) we can write the terms on the left-hand side of (7.75) as follows:

$$\frac{1}{2\pi}\int_0^{2\pi} \ln^+ \frac{1}{|f(Re^{i\alpha})-a|}\,d\alpha = m\left(R, \frac{1}{f-a}\right) = m(R, a),$$

$$\int_0^R \frac{n(t, a)-n(0, a)}{t}\,dt + n(0, a)\ln R = N\left(R, \frac{1}{f-a}\right) = N(R, a).$$

We can now write Nevanlinna's first fundamental theorem in the short form

$$\begin{aligned} m(R, a) + N(R, a) &= m(R, \infty) + N(R, \infty) + O(1) \\ &= T(R) + O(1). \end{aligned} \qquad (7.76)$$

To illustrate these ideas we use the simple example of $f(z) = e^z$. Since in this case $n(t, \infty) \equiv 0$, we have

$$\begin{aligned} T(R) &= \frac{1}{2\pi}\int_0^{2\pi} \ln^+ |e^{Re^{i\alpha}}|\,d\alpha = \frac{1}{2\pi}\int_0^{2\pi} \ln^+ e^{R\cos\alpha}\,d\alpha \\ &= \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} \ln^+ e^{R\cos\alpha} = \frac{R}{2\pi}\int_{-\pi/2}^{\pi/2} \cos\alpha\,d\alpha = \frac{R}{\pi}. \end{aligned}$$

We put $a = 0$; since $n(t, 0) \equiv 0$, we find that

$$\begin{aligned} m(R, 0) + N(R, 0) &= \frac{1}{2\pi}\int_0^{2\pi} \ln^+ \left|\frac{1}{e^{Re^{i\alpha}}}\right| d\alpha \\ &= \frac{1}{2\pi}\int_{\pi/2}^{3\pi/2} \ln^+ e^{-R\cos\alpha}\,d\alpha = T(R) = \frac{R}{\pi}. \end{aligned}$$

If we take $a \neq 0$, then $n(t, a)$ is the number of roots of the equation $e^z = a$ that lie in the circle $|z| \leqslant t$. These roots, which are equal to $\operatorname{Ln} a = \ln|a| + i \arg a + 2k\pi i$, lie on the straight line $x = \ln|a|$ at a distance 2π from each other. Whence

$$n(t, a) \sim \frac{2\sqrt{t^2-(\ln|a|)^2}}{2\pi} \sim \frac{t}{\pi} \text{ and } N(R, a) \sim \int_0^R \frac{t}{\pi t}\,dt = \frac{1}{\pi}R.$$

As to $m(R, a) = \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} \ln^+ \frac{1}{|e^{Re^{i\alpha}} - a|}\,d\alpha$, it remains bounded (one must separately consider $Re^{i\alpha}$ in the right half-plane, where

$e^{Re^{i\alpha}} \to \infty$ inside any angle $-\pi/2 + \varepsilon < \alpha < \pi/2 - \varepsilon$, and in the left half-plane, where $e^{Re^{i\alpha}} \to 0$ inside any angle $\pi/2 + \varepsilon < \alpha < 3\pi/2 - \varepsilon$; $\varepsilon > 0$). Hence, when $a \neq 0$, we also find that $N(R, a) + m(R, a) \approx R/\pi$. This example shows clearly that although e^z vanishes nowhere, zero is a good approximation to e^z in the mean on the circle $|z| = R$. When a is nonzero, $m(R, a)$ remains bounded, i.e. a is a poor approximation to e^z in the mean; in this case the second term in $m(R, a) + N(R, a)$ plays the main role, and this term depends on how often e^z is exactly a ($N(R, a) \approx R/\pi$).

8

RESIDUES AND THEIR APPLICATIONS. THE ARGUMENT PRINCIPLE. ELLIPTIC FUNCTIONS

8.1

THE RESIDUE THEOREM AND THE EVALUATION OF DEFINITE INTEGRALS

Here we will examine the evaluation of integrals of single-valued analytic functions along closed curves. We assume all along that in a region that contains the integration contour there are no singularities except isolated singularities. On this assumption, the interior of the contour may contain only a finite number of singular points (otherwise there would be at least one limit point of the singular points that is a nonisolated singularity of the function). Suppose that $z_1, z_2, \ldots, z_n$ are the isolated singularities of $f(z)$ that lie in the interior of a closed rectifiable curve Γ. Around each point z_k we draw a circle γ_k: $|z - z_k| = \rho_k$ of such a small radius ρ_k that this circle lies inside Γ and outside all the other circles. Then in view of the composite contour theorem we have

$$\int_{\Gamma} f(z)\,dz = \int_{\gamma_1} f(z)\,dz + \int_{\gamma_2} f(z)\,dz + \ldots + \int_{\gamma_n} f(z)\,dz.$$

Hence, we need to evaluate the integral $\int_{\gamma_k} f(z)\,dz$ along the circle $|z - z_k| = \rho_k$ that lies in a (small) neighborhood of the isolated singularity z_k of $f(z)$.

We use the Laurent expansion of $f(z)$ at z_k and integrate termwise (which is possible due to the uniform convergence of the Laurent series on γ_k). This yields

$$\int_{\gamma_k} f(z)\,dz = \int_{\gamma_k} \sum_{m=-\infty}^{+\infty} a_m^{(k)} (z - z_k)^m\,dz$$

$$= \sum_{m=-\infty}^{+\infty} a_m^{(k)} \int_{\gamma_k} (z - z_k)^m\,dz = a_{-1}^{(k)}\,2\pi i.$$

Indeed, among the integrals $\int\limits_{\gamma_k} (z-z_k)^m\, dz$ only the one with $m=-1$ is not zero, and it is equal to $2\pi i$.

Thus

$$\int\limits_{\Gamma} f(z)\, dz = 2\pi i\, (a_{-1}^{(1)} + a_{-1}^{(2)} + \ldots + a_{-1}^{(n)}). \qquad (8.1)$$

Formula (8.1) provides a complete solution to the problem. We see that on the assumptions we have made the value of the integral of an analytic function depends only on the coefficients a_{-1} of the $(z-z_k)^{-1}$ in the Laurent expansions of $f(z)$ at the isolated singularities that lie inside the integration contour. These coefficients are called the *residues* of the function.*

In other words, *the residue of $f(z)$ at an isolated singularity a is the coefficient c_{-1} of $(z-a)^{-1}$ in the Laurent expansion of $f(z)$ at this point.*

Formula (8.1) is the manifestation of

The residue theorem. *The integral of a function $f(z)$ along a closed contour Γ that lies in a domain where the function is single-valued and analytic except at isolated singularities and that does not pass through any such points is $2\pi i$ times the sum of the residues of $f(z)$ at the isolated singularities that lie inside Γ.*

To apply this theorem we must learn to find the residues. These are easily calculated if the isolated singularities are poles. Suppose that a is a simple pole of the function. Then at a the Laurent expansion is

$$f(z) = \frac{a_{-1}}{z-a} + a_0 + a_1(z-a) + \ldots,$$

---

* The idea of the residue belongs to Cauchy. He also discovered the various applications of this idea to various problems in calculus. The term residue originated, apparently, from the fact that Cauchy came to this idea in trying to find the difference between integrals taken along curves that have a common beginning and common end and between which lie the poles of the function. In this form the idea of residues can be observed in his paper "Mémoire sur la theórie des intégrales définies" (1814). The term *residue* was first used in his paper "Sur un nouveau genre de calcul (Analogue au calcul infinitésimal)" in the first volume of his *Exercices de mathématique* (1826). Here is how Cauchy introduces this concept: "If after we have found the values of x that turn $f(x)$ into infinity we add to one of these values, designated x_1, an infinitesimal quantity ε and then expand in a series the function $f(x_1+\varepsilon)$ in increasing powers of the same quantity, the first terms in the expansion will contain negative powers of ε and one of them will be the product of $1/\varepsilon$ into a finite coefficient, which we call the residue [le résidu] of the function $f(x)$ at the particular value x_1 of the variable x." In a number of papers published after this one in this and the following three volumes of *Exercices* (1826-9), Cauchy considers the application of his method to the evaluation of integrals, expansion of functions in series and infinite products, equation theory, etc.

whence

$$f(z)(z-a) = a_{-1} + a_0(z-a) + a_1(z-a)^2 + \ldots$$

and, therefore,

$$a_{-1} = \operatorname*{Res}_{z=a} f(z) = \lim_{z \to a} [f(z)(z-a)]. \tag{8.2}$$

The calculation of a residue is simplified still further if $f(z)$ can be represented in the form of a quotient:

$$f(z) = \frac{\varphi(z)}{\psi(z)},$$

where $\varphi(a) \neq 0$, and $\psi(z)$ has a simple zero at $z = a$ (i.e. $\psi(a) = 0$ and $\psi'(a) \neq 0$). Then $z = a$ is a simple pole of $f(z)$ and (8.2) yields

$$\operatorname*{Res}_{z=a} f(z) = \operatorname*{Res}_{z=a} \frac{\varphi(z)}{\psi(z)} = \lim_{z \to a} \frac{\varphi(z)(z-a)}{\psi(z)}$$

$$= \lim_{z \to a} \frac{\varphi(z)}{\dfrac{\psi(z)-\psi(a)}{z-a}} = \frac{\varphi(a)}{\psi'(a)}. \tag{8.3}$$

If a is a pole of the kth order ($k > 1$), at a we have the following expansion:

$$f(z) = \frac{a_{-k}}{(z-a)^k} + \ldots + \frac{a_{-1}}{z-a} + a_0 + a_1(z-a) + \ldots,$$

whence

$$f(z)(z-a)^k = a_{-k} + a_{-k+1}(z-a) + \ldots + a_{-1}(z-a)^{k-1} + \ldots.$$

If we differentiate termwise $k-1$ times, we have

$$\frac{d^{k-1}[f(z)(z-a)^k]}{dz^{k-1}} = (k-1)!\, a_{-1} + k(k-1) \ldots 2a_0(z-a) + \ldots$$

and, finally, as $z \to a$,

$$(k-1)!\, a_{-1} = \lim_{z \to a} \frac{d^{k-1}[f(z)(z-a)^k]}{dz^{k-1}},$$

or

$$a_{-1} = \operatorname*{Res}_{z=a} f(z) = \frac{1}{(k-1)!} \lim_{z \to a} \frac{d^{k-1}[f(z)(z-a)^k]}{dz^{k-1}}. \tag{8.4}$$

Examples. (a) Evaluate the integral $\int\limits_{-\infty}^{+\infty} F(x)\, dx$, where $F(x)$ is a rational function equal to $P(x)/Q(x)$ that has no poles on the real axis and is such that the degree of the denominator $Q(x)$ is higher by at least two units than the degree of the numerator $P(x)$. We take

the integration contour depicted in Fig. 54, where BCA is a semicircle of radius R centered at the origin of coordinates. We select R so large that all the poles of $F(z)$ in the upper half-plane lie inside this contour. Then we have

$$\int_{-R}^{+R} F(x)\,dx + \int_{BCA} F(z)\,dz = 2\pi i \sum \operatorname{Res} F(z),$$

where the sum is over all poles of $F(z)$ in the upper half-plane. Since

$$|F(z)| = \left|\frac{P(z)}{Q(z)}\right| = \left|\frac{a_m z^m + \ldots + a_0}{b_n z^n + \ldots + b_0}\right| = \left|\frac{a_m}{b_n z^{n-m}}\right| \left|\frac{1 + \ldots + \dfrac{a_0}{a_m z^m}}{1 + \ldots + \dfrac{b_0}{b_n z^n}}\right|$$

and $n - m \geqslant 2$, for sufficiently large values of $|z| = R$ we have

$$|F(z)| < \frac{2\,|a_m|}{|b_n|\,R^2} = \frac{C}{R^2}.$$

Whence

$$\left|\int_{BCA} F(z)\,dz\right| < \frac{C}{R^2}\pi R = \frac{\pi C}{R} \to 0 \text{ as } R \to \infty.$$

Therefore,

$$\int_{-\infty}^{+\infty} F(x)\,dx = \lim_{R\to\infty} \int_{-R}^{+R} F(x)\,dx = 2\pi i \sum \operatorname{Res} F(z).$$

Thus, the integral of a rational function that has no poles on the real axis and has at the point at infinity a zero at least of the second order (this is equivalent to requiring that $n - m \geqslant 2$) is equal to $2\pi i$ times the sum of the residues of $F(z)$ at the poles in the upper half-plane.

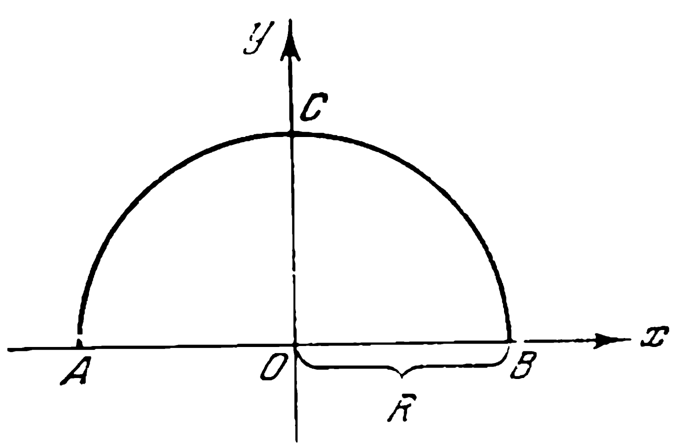


Fig. 54

Suppose that

$$F(z) = \frac{z^{2p} - z^{2q}}{1 - z^{2r}},$$

where p, q, and r are nonnegative integers, and $p < r$ and $q < r$.

The degree $2r$ of the denominator is at least two units higher than that of the numerator. All poles of $F(z)$ are expressed by the formula

$$z = e^{\frac{k\pi i}{r}} \qquad (k = 1, \ldots, r-1, r+1, \ldots, 2r-1)$$

(points $+1$ and -1 are not poles of $F(z)$ since the numerator and denominator have a common factor $1 - z^2$; if $p - q$ and r are not

relatively prime numbers, then some of the above points will also not be poles of $F(z)$). Among these the poles in the upper half-plane are

$$z=e^{\frac{k\pi i}{r}} \qquad (k=1,\,2,\,\ldots,\,r-1).$$

All these poles are simple and, hence,

$$\operatorname*{Res}_{z=e^{\frac{k\pi i}{r}}} F(z)=\frac{e^{\frac{k\pi i}{r}2p}-e^{\frac{k\pi i}{r}2q}}{-2re^{\frac{k\pi i}{r}(2r-1)}}=\frac{1}{2r}\left[e^{(2q+1)\frac{k\pi i}{r}}-e^{(2p+1)\frac{k\pi i}{r}}\right].$$

Whence

$$\int_{-\infty}^{+\infty}\frac{x^{2p}-x^{2q}}{1-x^{2r}}\,dx=\frac{\pi i}{r}\sum_{1}^{r-1}\left[e^{(2q+1)\frac{k\pi i}{r}}-e^{(2p+1)\frac{k\pi i}{r}}\right]$$

$$=\frac{\pi}{r}\left[i\,\frac{1+e^{(2q+1)\frac{\pi i}{r}}}{1-e^{(2q+1)\frac{\pi i}{r}}}-i\,\frac{1+e^{(2p+1)\frac{\pi i}{r}}}{1-e^{(2p+1)\frac{\pi i}{r}}}\right]$$

$$=\frac{\pi}{r}\left(\cot\frac{2p+1}{2r}\pi-\cot\frac{2q+1}{2r}\pi\right).$$

Since the integrand is an even function,

$$\int_{0}^{\infty}\frac{x^{2p}-x^{2q}}{1-x^{2r}}\,dx=\frac{\pi}{2r}\left(\cot\frac{2p+1}{2r}\pi-\cot\frac{2q+1}{2r}\pi\right).$$

If $r=2n$ and $q=p+n$ $(p<n)$, this formula becomes

$$\int_{0}^{\infty}\frac{x^{2p}}{1+x^{2n}}\,dx=\frac{\pi}{2n\sin\frac{2p+1}{2n}\pi}.$$

(b) **Jordan's lemma.** *Let $\Phi(z)$ be analytic, except for a finite number of poles, in the upper half-plane and on the real axis (excluding the point at infinity). If it has no poles on the real axis and tends to zero as z tends to ∞ in the closed upper half-plane, then for any positive μ we have the following formula:*

$$\int_{0}^{\infty}\left[e^{\mu ix}\Phi(x)+e^{-\mu ix}\Phi(-x)\right]dx=2\pi i\sum\operatorname{Res}\left[e^{\mu iz}\Phi(z)\right],$$

where the sum includes all the poles of $\Phi(z)$ in the upper half-plane.

If we take the same integration contour as in example (a), we have

$$\int_{-R}^{R} e^{\mu i x}\Phi(x)\,dx + \int_{BCA} e^{\mu i z}\,\Phi(z)\,dz = 2\pi i \sum \operatorname{Res}\,[e^{\mu i z}\Phi(z)].$$

We select R so large that all poles of the function $e^{\mu i z}\Phi(z)$ in the upper half-plane (they coincide with the poles of $\Phi(z)$) lie inside the integration contour. Let us show that in such conditions

$$\lim_{R\to\infty} \int_{BCA} e^{\mu i z}\Phi(z)\,dz = 0.$$

Indeed,

$$|J_R| = \left| \int_{BCA} e^{\mu i z}\Phi(z)\,dz \right|$$

$$= \left| \int_0^{\pi} \exp(\mu i R\cos\varphi - \mu R\sin\varphi)\,\Phi(Re^{i\varphi})\,iRe^{i\varphi}\,d\varphi \right|$$

$$\leqslant \int_0^{\pi} e^{-\mu R\sin\varphi}\,|\Phi(Re^{i\varphi})|\,R\,d\varphi.$$

Since we have assumed that $\max\limits_{0\leqslant\varphi\leqslant\pi} |\Phi(Re^{i\varphi})| = \varepsilon(R) \to 0$ as $R\to\infty$, we have

$$|J_R| < \varepsilon(R)\int_0^{\pi} Re^{-\mu R\sin\varphi}\,d\varphi = 2\varepsilon(R)\int_0^{\pi/2} Re^{-\mu R\sin\varphi}\,d\varphi$$

$$< 2\varepsilon(R)\int_0^{\pi/2} Re^{-\frac{2}{\pi}\mu R\varphi}\,d\varphi = \frac{\pi\varepsilon(R)}{\mu}(1-e^{-\mu R}) < \frac{\pi\varepsilon(R)}{\mu} \to 0$$

as $R\to\infty$.

Hence,

$$\lim_{R\to\infty}\int_{-R}^{+R} e^{\mu i x}\Phi(x)\,dx = \lim_{R\to\infty}\int_0^{R} [e^{\mu i x}\Phi(x) + e^{-\mu i x}\Phi(-x)]\,dx$$

$$= \int_0^{\infty} [e^{\mu i x}\Phi(x) + e^{-\mu i x}\Phi(-x)]\,dx = 2\pi i \sum \operatorname{Res}\,[e^{\mu i z}\Phi(z)].$$

This is the sought for result. If $\Phi(z)$ is an even function,

$$\int_0^{\infty} \cos\mu x\,\Phi(x)\,dx = \pi i \sum \operatorname{Res}\,[e^{\mu i z}\Phi(z)],$$

and if $\Phi(z)$ is an odd function,

$$\int_0^\infty \sin\mu x\Phi(x)\,dx = \pi\sum \operatorname{Res}\,[e^{\mu iz}\Phi(z)].$$

For instance,

$$\int_0^\infty \frac{\cos\mu x}{a^2+x^2}\,dx = \pi i \operatorname*{Res}_{z=ai}\frac{e^{\mu iz}}{a^2+z^2} = \frac{\pi e^{-\mu a}}{2a},$$

$$\int_0^\infty \frac{x\sin\mu x}{a^2+x^2}\,dx = \pi \operatorname*{Res}_{z=ai}\frac{ze^{\mu iz}}{a^2+z^2} = \frac{\pi e^{-\mu a}}{2}.$$

8.2

THE ARGUMENT PRINCIPLE AND ITS IMPLICATIONS

As an important example of the applications of the theory of residues we will evaluate the integral $\frac{1}{2\pi i}\int\limits_{\Gamma}\varphi(z)\frac{f'(z)}{f(z)-A}\,dz$, where $f(z)$ is a function that is single-valued in a domain G and has in it no singularities except, perhaps, poles, A is an arbitrary complex number, $\varphi(z)$ a function that is single-valued and analytic in the same domain, and Γ a rectifiable Jordan curve that belongs to G together with its interior and passes through neither the poles nor the A-points of $f(z)$.

The singularities of $F(z) = \varphi(z)\frac{f'(z)}{f(z)-A}$ in G may only be poles that originate from the A-points or poles of $f(z)$. Suppose that $a_1, \ldots, a_m$ are the A-points of $f(z)$ in the interior of Γ, with $\alpha_1, \ldots, \alpha_m$ the orders of these A-points, and $b_1, \ldots, b_n$ are the poles of $f(z)$ in the interior of Γ, with $\beta_1, \ldots, \beta_n$ the orders of these poles. In a neighborhood of point a_j the functions $\varphi(z)$ and $f(z)$ have the following expansions:

$$\varphi(z) = \varphi(a_j) + \ldots, \quad f(z) - A = c_{\alpha_j}(z-a_j)^{\alpha_j} + \ldots .$$

Hence, $f'(z) = c_{\alpha_j}\alpha_j(z-a_j)^{\alpha_j-1} + \ldots$ and

$$F(z) = [\varphi(a_j)+\ldots]\frac{c_{\alpha_j}\alpha_j(z-a_j)^{\alpha_j-1}+\ldots}{c_{\alpha_j}(z-a_j)^{\alpha_j}+\ldots}$$

$$= \frac{\alpha_j}{z-a_j}[\varphi(a_j)+\ldots]\frac{1+\ldots}{1+\ldots}$$

$$= \frac{\alpha_j}{z-a_j}[\varphi(a_j)+\ldots] = \frac{\alpha_j\varphi(a_j)}{z-a_j}+\ldots .$$

The terms we omitted contain higher powers of $z - a_j$. For instance, the term that follows the one with $(z - a_j)^{-1}$ is the absolute term of the Laurent expansion, the one after that contains $z - a_j$, etc. This implies that $z = a_j$ is a simple pole of $F(z)$ with the residue equal to $\alpha_j \varphi(a_j)$. This residue is zero if $\varphi(a_j) = 0$; in this case a_j is not a pole of $F(z)$.

Let us turn now to one of the poles b_j of $f(z)$. The expansions at this point are

$$\varphi(z) = \varphi(b_j) + \ldots,$$
$$f(z) - A = d_{-\beta_j}(z - b_j)^{-\beta_j} + \ldots,$$
$$f'(z) = -\beta_j d_{-\beta_j}(z - b_j)^{-\beta_j - 1} - \ldots,$$

whence

$$F(z) = [\varphi(b_j) + \ldots] \frac{-\beta_j d_{-\beta_j}(z-b_j)^{-\beta_j-1} - \ldots}{d_{-\beta_j}(z-b_j)^{-\beta_j} + \ldots}$$
$$= \frac{-\beta_j}{z-b_j}[\varphi(b_j) + \ldots] \frac{1+\ldots}{1+\ldots}$$
$$= -\frac{\beta_j}{z-b_j}[\varphi(b_j) + \ldots] = -\frac{\beta_j \varphi(b_j)}{z-b_j} - \ldots.$$

Hence, $F(z)$ has a simple pole at $z = b_j$ with the residue equal to $-\beta_j \varphi(b_j)$ (which is zero if $\varphi(b_j) = 0$).

Applying the residue theorem to the integral

$$\frac{1}{2\pi i} \int_\Gamma \varphi(z) \frac{f'(z)}{f(z) - A} dz,$$

we obtain

$$\frac{1}{2\pi i} \int_\Gamma \varphi(z) \frac{f'(z)\,dz}{f(z) - A} = \sum_{j=1}^{m} \alpha_j \varphi(a_j) - \sum_{j=1}^{n} \beta_j \varphi(b_j). \qquad (8.5)$$

The first sum on the right-hand side is the sum of values that $\varphi(z)$ admits at the A-points of $f(z)$, and each term is repeated as many times as the order of the respective A-point. If we assume that each A-point in the interior of Γ is included in the sum as many times as its order, then $\sum_{j=1}^{m} \alpha_j \varphi(a_j)$ can simply be said to be the sum of values of $\varphi(z)$ at the A-points of $f(z)$. A similar statement can be made for the second sum, where the summation is over the poles of $f(z)$. We arrive at the following statement.

The integral $\frac{1}{2\pi i} \int_\Gamma \varphi(z) \frac{f'(z)}{f(z) - A} dz$ *is equal to the difference between the sum of values that* $\varphi(z)$ *admits at the* A*-points of* $f(z)$ *lying in*

the interior of Γ *and the sum of values that* $\varphi(z)$ *admits at the poles of* $f(z)$ *lying in the interior of* Γ.

Here are two special cases of the above proposition.

(a) $\varphi(z) = z$. We have

$$\frac{1}{2\pi i}\int_{\Gamma} z\frac{f'(z)}{f(z)-A}\,dz=\sum_{j=1}^{m}\alpha_j a_j-\sum_{j=1}^{n}\beta_j b_j, \tag{8.6}$$

i.e. the integral is equal to the difference between the sum of A-points of $f(z)$ inside Γ and the sum of poles of this function inside Γ.

(b) $\varphi(z) \equiv 1$. We have

$$\frac{1}{2\pi i}\int_{\Gamma} \frac{f'(z)}{f(z)-A}\,dz=\sum_{j=1}^{m}\alpha_j-\sum_{j=1}^{n}\beta_j, \tag{8.7}$$

i.e. the integral is equal to the difference between the number of A-points of $f(z)$ inside Γ and the number of poles inside Γ.

If $A = 0$, then the A-points are the zeros of $f(z)$. If their number inside Γ is N and the number of poles inside Γ is P,

$$\frac{1}{2\pi i}\int_{\Gamma} \frac{f'(z)}{f(z)}\,dz=N-P. \tag{8.8}$$

The integral on the left-hand side is known as the *logarithmic residue* of $f(z)$ with respect to a contour Γ (we note that the integrand is the logarithmic derivative of $f(z)$). We come to the following

Theorem. *The difference between the zeros and poles of a function* $f(z)$ *inside a contour* Γ *is equal to the logarithmic residue of the same function with respect to the contour (both quantities are calculated with due regard for the orders of the poles and zeros).*

The meaning of the logarithmic residue is simple. To reveal it we write the integral in the form

$$\frac{1}{2\pi i}\int_{\Gamma} \frac{f'(z)}{f(z)}\,dz=\frac{1}{2\pi i}\int_{\Gamma}\frac{d}{dz}\{\mathrm{Ln}\,[f(z)]\}\,dz.$$

On Γ we take an arbitrary point z_0, which we assume to be the initial and final point of integration. When a point z traverses Γ in the positive sense, $\mathrm{Ln}\, f(z)$ changes continuously and after the curve has been traversed completely, its value at z_0, generally speaking, differs from the initial value at the same point. But for the same $f(z_0)$ the values of $\mathrm{Ln}\, f(z_0)$ may differ only due to the different values that $\mathrm{Arg}\, f(z_0)$ admits before and after the traversal. If we denote the initial value of $\mathrm{Arg}\, f(z_0)$ by Φ_0 and the value of $\mathrm{Arg}\, f(z_0)$

after the traversal by Φ_1, we find that

$$\frac{1}{2\pi i}\int_{\Gamma}\frac{f'(z)}{f(z)}\,dz=\frac{1}{2\pi i}\{[\ln|f(z_0)|+i\Phi_1]-[\ln|f(z_0)|+i\Phi_0]\}=\frac{\Phi_1-\Phi_0}{2\pi}.$$

Hence, by (8.8),

$$N-P=\frac{\Phi_1-\Phi_0}{2\pi} \tag{8.9}$$

or, if we denote $\Phi_1-\Phi_0$ by $\operatorname{Var}_{\Gamma}\operatorname{Arg} f(z)$ (where Var stands for variation),

$$N-P=\frac{1}{2\pi}\operatorname{Var}_{\Gamma}\operatorname{Arg} f(z).$$

The above relationship expresses

The argument principle. *The difference between the number of zeros and poles of $f(z)$ that lie inside a closed curve Γ is equal to the variation of* Arg $f(z)$ *when point z traverses Γ in the positive sense, divided by 2π.*

We also note the geometrical meaning of the above proposition. As point z traverses the closed curve Γ in the positive sense, the terminal point of vector $w=f(z)$ traverses a closed curve Γ'. We denote the number of complete revolutions about the origin of coordinates as vector w completes one revolution in the course of such a traversal by ν. A revolution is regarded as $+1$ if it proceeds in the positive sense and as -1 if in the negative. Then the variation of Arg $f(z)$ is simply $2\pi\nu$, which leads to the following formulation of the argument principle:

The difference between the number of zeros and poles of a single-valued function $f(z)$ that lie in the interior of a closed curve Γ is equal to the number of revolutions ν about the origin of coordinates which the vector $f(z)$ completes as point z traverses Γ in the positive sense.

In the particular case where $f(z)$ has no poles in the interior of Γ we arrive at the following proposition:

The number of zeros of $f(z)$ that lie in the interior of a closed curve Γ is equal to the number of revolutions about the origin of coordinates which the vector $f(z)$ completes as point z traverses Γ in the positive sense.

The argument principle leads to

Rouche's theorem (1862). *Let $f(z)$ and $\varphi(z)$ be functions single-valued and analytic at points on a closed rectifiable curve Γ and in its interior. Suppose that $|f(z)|>|\varphi(z)|$ on Γ. Then the number of zeros of $f(z)+\varphi(z)$ in the interior of Γ is equal to that of $f(z)$.*

Proof. To find the number of zeros of $f(z)+\varphi(z)$ we use the argument principle. If we write this sum for points on Γ as

$$f(z)+\varphi(z)=f(z)\left[1+\frac{\varphi(z)}{f(z)}\right]$$

($|f(z)|$ on Γ is greater than $|\varphi(z)|$ and, therefore, does not vanish) we find that

$$\operatorname{Arg}[f(z)+\varphi(z)] = \operatorname{Arg} f(z) + \operatorname{Arg}\left[1+\frac{\varphi(z)}{f(z)}\right].$$

But $|\varphi(z)/f(z)| < 1$. Hence, the terminal point of vector $1 + \varphi(z)/f(z)$ traverses a closed curve that lies completely in a circle with a center at 1 and a radius equal to 1. Consequently, the corresponding vector does not make a single revolution about the origin of coordinates and the variation of $\operatorname{Arg}\left[1+\frac{\varphi(z)}{f(z)}\right]$ when point z traverses Γ is zero. Thus, the variation of $\operatorname{Arg}[f(z)+\varphi(z)]$ in the above traversal coincides with that of $\operatorname{Arg} f(z)$ in the same traversal, whereby according to the argument principle the number of zeros of $f(z)$ and $f(z)+\varphi(z)$ are the same.

A useful corollary of this theorem is the

Hurwitz's theorem (1889). *If $\{f_n(z)\}$ is a sequence of functions analytic in a domain G and if this sequence converges uniformly inside the domain to a function $f(z) \not\equiv 0$, then for any closed rectifiable curve γ that belongs to G together with its interior and does not pass through the zeros of $f(z)$ there exists a number $\nu = \nu(\gamma)$ such that for $n > \nu(\gamma)$ each function $f_n(z)$ has the same amount of zeros inside γ equal to the number of zeros of $f(z)$ lying inside γ.*

Proof. Suppose that μ is the minimum of $|f(z)|$ on γ. By virtue of the hypothesis, $\mu > 0$. Hence, in view of the uniform convergence of $\{f_n(z)\}$, on γ we can indicate a $\nu(\gamma)$ such that for $n > \nu(\gamma)$ the inequality

$$|f_n(z) - f(z)| < \mu \leqslant |f(z)|$$

is valid for all points of γ. Whence, by Rouche's theorem the functions $f(z)$ and $f(z) + [f_n(z) - f(z)] = f_n(z)$ $(n > \nu(\gamma))$ have the same number of zeros in the interior of γ. This completes the proof.

Examples. (a) Find the number of roots of the equation $z^8 - 4z^5 + z^2 - 1 = 0$ whose moduli are less than unity.

We apply Rouche's theorem. To this end we write $z^8 - 4z^5 + z^2 - 1$ as

$$f(z) + \varphi(z), \text{ where } f(z) = -4z^5 \text{ and } \varphi(z) = z^8 + z^2 - 1.$$

Since for $|z| = 1$,

$$|\varphi(z)| = |z^8 + z^2 - 1| \leqslant |z^8| + |z^2| + 1 = 3$$

and

$$|f(z)| = |4z^5| = 4,$$

we find that

$$|\varphi(z)| < |f(z)|.$$

Hence, by Rouche's theorem the function $f(z) + \varphi(z) = z^8 - 4z^5 + z^2 - 1$ has inside the circle $|z| = 1$ as many zeros as the func-

tion $f(z) = -4z^5$. But the latter has a zero of the fifth order at the origin of coordinates; hence, it has five zeros inside the unit circle. For this reason, the equation $z^8 - 4z^5 + z^2 - 1 = 0$ has five roots inside the unit circle, i.e. five roots whose moduli are less than unity.

(b) Show that the equation

$$a_0 + a_1 \cos \vartheta + a_2 \cos 2\vartheta + \ldots + a_n \cos n\vartheta = 0,$$

where $0 < a_0 < a_1 < \ldots < a_n$, has $2n$ distinct roots in the interval $0 < \vartheta < 2\pi$ and all of them are real.

First we will prove that the zeros of the polynomial $p(z) = a_0 + a_1 z + \ldots + a_n z^n$ all lie inside the unit circle. Obviously, the polynomial has no positive real roots. But if z is not positive, then

$$\begin{aligned} |p(z)(z-1)| &= |a_n z^{n+1} - [a_0 + (a_1 - a_0) z + \ldots \\ &\qquad \ldots + (a_n - a_{n-1}) z^n]| \\ &\geqslant |a_n z^{n+1}| - |a_0 + (a_1 - a_0) z + \ldots \\ &\qquad \ldots + (a_n - a_{n-1}) z^n| \\ &> a_n |z|^{n+1} - [a_0 + (a_1 - a_0)|z| + \ldots \\ &\qquad \ldots + (a_n - a_{n-1}) |z|^n] \end{aligned}$$

This is true because $a_0, a_1 - a_0, \ldots, a_n - a_{n-1}$ are positive and z is not, which implies that the vectors $a_0, (a_1 - a_0) z, \ldots, (a_n - a_{n-1}) z^n$ cannot point in the same direction and, hence,

$$|a_0 + (a_1 - a_0) z + \ldots + (a_n - a_{n-1}) z^n| < a_0 + (a_1 - a_0)|z| + \ldots + (a_n - a_{n-1})|z|^n.$$

If, besides, $|z| \geqslant 1$, then

$$\begin{aligned} &a_0 + (a_1 - a_0)|z| + \ldots + (a_n - a_{n-1})|z|^n \\ &\leqslant a_0 |z|^{n+1} + (a_1 - a_0)|z|^{n+1} + \ldots + (a_n - a_{n-1})|z|^{n+1} \\ &= [a_0 + (a_1 - a_0) + \ldots + (a_n - a_{n-1})]|z|^{n+1} = a_n |z|^{n+1}. \end{aligned}$$

Thus, for z nonpositive and $|z| \geqslant 1$ we have

$$|p(z)(z-1)| > a_n |z|^{n+1} - a_n |z|^{n+1} = 0, \quad \text{i.e.} \quad p(z)(z-1) \neq 0.$$

But this means that for z nonpositive and not smaller than unity in modulus, $p(z) \neq 0$. This is true for positive z's as well; hence, $p(z)$ has no zeros either in the exterior of the unit circle or on it. Whence the n zeros of the polynomial $p(z)$ lie strictly inside the unit circle.

We make point z traverse the circle $|z| = 1$ in the positive sense. Then the vector that represents $p(z)$ must, by the argument principle, make as many revolutions about the origin of coordinates as there are zeros of $p(z)$, i.e. n. Since in each revolution the curve that the terminal point of the vector traverses intersects the imaginary axis two times at least (once from above and once from below), we have at least $2n$ such intersections. Each corresponds to a definite position of point z on the circle $|z| = 1$, i.e. a definite value of its argument ϑ, which varies in one revolution from 0 to 2π.

Therefore, we have at least $2n$ different values of ϑ in the interval from 0 to 2π for which the point corresponding to $p(z) = p(e^{i\vartheta})$ is on the imaginary axis. For each of these values,

$$\begin{aligned}\operatorname{Re}[p(e^{i\vartheta})] &= \operatorname{Re}(a_0 + a_1e^{i\vartheta} + \ldots + a_ne^{in\vartheta})\\ &= \operatorname{Re}[a_0 + a_1(\cos\vartheta + i\sin\vartheta) + \ldots\\ &\qquad\qquad \ldots + a_n(\cos n\vartheta + i\sin n\vartheta)]\\ &= a_0 + a_1\cos\vartheta + \ldots + a_n\cos n\vartheta\end{aligned}$$

vanishes; hence, there are at least $2n$ roots of the equation

$$a_0 + a_1\cos\vartheta + \ldots + a_n\cos n\vartheta = 0$$

in the interval $(0, 2\pi)$.

Let us show that the total number of roots in this interval is exactly $2n$. To this end we put $e^{i\vartheta} = \zeta$. We then have

$$\cos k\vartheta = \frac{e^{ik\vartheta} + e^{-ik\vartheta}}{2} = \frac{\zeta^k + \zeta^{-k}}{2},$$

and, therefore,

$$\begin{aligned}a_0 + a_1\cos\vartheta + \ldots + a_n\cos n\vartheta\\ = \frac{1}{2}\zeta^{-n}(a_n + a_{n-1}\zeta + \ldots + a_1\zeta^{n-1} + 2a_0\zeta^n\\ + a_1\zeta^{n+1} + \ldots + a_n\zeta^{2n}).\end{aligned}$$

If $\zeta_1, \zeta_2, \ldots, \zeta_{2n}$ are the zeros of the polynomial on the right-hand side, all the zeros of the trigonometric polynomial on the left-hand side satisfy the following relations:

$$e^{i\vartheta} = \zeta_j \quad (j = 1, 2, \ldots, 2n).$$

In the first place this implies that the number of distinct real zeros of the trigonometric polynomial does not exceed $2n$ in the interval $(0, 2\pi)$. But earlier we proved that there are at least $2n$ distinct real zeros of this polynomial in the interval $(0, 2\pi)$. Hence, there are exactly $2n$ such zeros. We note that the moduli of $\zeta_j = e^{i\vartheta_j}$ are all equal to unity. Hence, there cannot be a single imaginary zero among those of the given trigonometric polynomial.

8.3

THE RESIDUE AT THE POINT AT INFINITY

If $f(z)$ is single-valued and analytic in a neighborhood $|z| > R$ of the point at infinity (with the exception, perhaps, of the point itself), then the following expansion is valid in that neighborhood:

$$(z) = \ldots + A_{-m}z^{-m} + \ldots + A_{-1}z^{-1} + A_0 + A_1 z + \ldots + A_n z^n + \ldots .$$

We integrate $f(z)$ along a circle $C_\sigma: |z| = \sigma$, with $\sigma > R$, and we choose the sense of traversal in a way such that the neighborhood $|z| > \sigma$ of the point at infinity remains to the left. It is logical to consider this sense positive with respect to the traversal of C_σ with respect to the point at infinity; with respect to the interior of the circle $|z| < \sigma$, i.e. with respect to a neighborhood of the finite point $z = 0$, this sense will be negative. As a result of termwise integration of the Laurent series we have

$$\int_{C_\sigma} f(z)\,dz = A_{-1}(-2\pi i) = 2\pi i(-A_{-1}).$$

If we want the integral along a contour that circles the point at infinity to be equal to the residue at that point times $2\pi i$, we must introduce the following definition:

The residue at the point at infinity is $-a_{-1}$, where a_{-1} is the coefficient with index -1 of the Laurent expansion of $f(z)$ at ∞.

Then

$$\int_{C_\sigma} f(z)\,dz = 2\pi i \operatorname*{Res}_{z=\infty} f(z),$$

where the sense of traversal of C_σ is positive with respect to the point at infinity, i.e. the exterior of the contour is always to the left instead of the interior, as is usually the case.

Using this definition, we can formulate the following

Theorem. *The sum of all residues of a single-valued analytic function that has only isolated singularities in the extended plane is zero.*

Indeed, we note, first, that such a function has only a finite number of singularities (otherwise there would exist a finite or infinite limit point for the set of singular points, which would therefore be a nonisolated singular point of the function). We take a circle $|z| = \sigma$ centered at the origin of coordinates such that on it and in its exterior (except, perhaps, at point $z = \infty$) there are no singularities of the function. Then all the finite points $z_1, z_2, \ldots, z_n$ will lie

inside this circle, so that by the residue theorem we have

$$\int_{C_\sigma} f(z)\,dz = 2\pi i \sum_{l=1}^{n} \operatorname*{Res}_{z=z_k} f(z).$$

Here C_σ is traversed in the usual positive sense, i.e. the interior of C_σ remains to the left. But the same sense is negative with respect to the point at infinity, whereby

$$\int_{C_\sigma} f(z)\,dz = -2\pi i \operatorname*{Res}_{z=\infty} f(z).$$

If now we subtract the second expression from the first, we find that

$$2\pi i\,[\operatorname*{Res}_{z=z_1} f(z) + \ldots + \operatorname*{Res}_{z=z_n} f(z) + \operatorname*{Res}_{z=\infty} f(z)] = 0,$$

or

$$\operatorname*{Res}_{z=z_1} f(z) + \ldots + \operatorname*{Res}_{z=z_n} f(z) + \operatorname*{Res}_{z=\infty} f(z) = 0.$$

The proof of the theorem is complete.

For one, the theorem is valid for any rational function, since a rational function can have only isolated singularities (namely, poles).

We note that the residue at the point at infinity is determined by the coefficient of a term in the regular part of the Laurent expansion, whereas the residue at a finite point is determined by the coefficient of a term in the principal part (the reader will recall that in a Laurent expansion the terms with negative powers constitute the regular part for the point $z = \infty$ and the principal part for a finite point). This implies that the residue at point $z = \infty$ may differ from zero even when this point is not a singular point, i.e. is regular, whereas the residue at a finite regular point is always zero. For instance, for $f(z) = 1/z$ the point $z = \infty$ is regular (a zero of order 1), but $\operatorname*{Res}_{z=\infty} f(z) = -1 \neq 0$.

8.4

THE PARTIAL-FRACTION EXPANSION OF MEROMORPHIC FUNCTIONS BY MEANS OF THE RESIDUE THEOREM

Suppose that $f(z)$ is a single-valued analytic function with no singular points in the finite complex plane except poles.

We denote a rectifiable Jordan curve that does not pass through the poles of $f(z)$ by C and let z be a point in the interior of C that is not the origin of coordinates or a pole. We wish to evaluate the

integral

$$\frac{1}{2\pi i}\int_C \frac{f(\zeta)\,d\zeta}{\zeta-z}.$$

This is obviously an integral of the Cauchy type (see Sec. 6.10).

The poles of the function $\varphi(\zeta) = f(\zeta)/(\zeta - z)$ lying in the interior of C are the point $\zeta = z$ and all the poles of $f(z)$ that lie in the interior of C. We denote those of the poles that are nonzero by $\beta_1, \ldots, \beta_n$ and the principal parts of the corresponding Laurent expansions of $f(z)$ by $G_1(z), \ldots, G_n(z)$. In addition we put $\beta_0 = 0$ and assume that $G_0(z)$ is the principal part of the Laurent expansion of $f(z)$ at point $z = 0$, so that $G_0(z)$ is identically zero if $z = 0$ is a regular point for $f(z)$ and is a rational function with only one pole at the origin of coordinates if $z = 0$ is a pole of $f(z)$.

Let us now calculate the residues of $\varphi(\zeta)$ at $\zeta = z, \beta_0, \ldots, \beta_n$. First we find that

$$\operatorname*{Res}_{\zeta=z}\varphi(\zeta) = f(z).$$

Next, if go over from $f(\zeta)$ to its Laurent expansion at β_k,

$$f(\zeta) = \frac{A^{(k)}_{-\gamma_k}}{(\zeta-\beta_k)^{\gamma_k}} + \ldots + \frac{A^{(k)}_{-1}}{\zeta-\beta_k} + A^{(k)}_0 + A^{(k)}_1(\zeta-\beta_k) + \ldots$$

$$= G_k(\zeta) + P_k(\zeta),$$

where $G_k(\zeta)$ and $P_k(\zeta)$ are the principal and regular parts of the expansion, respectively, and note that for $|\zeta - \beta_k| < |z - \beta_k|$,

$$\frac{1}{\zeta-z} = -\frac{1}{(z-\beta_k)-(\zeta-\beta_k)}$$

$$= -\frac{1}{z-\beta_k} - \frac{\zeta-\beta_k}{(z-\beta_k)^2} - \ldots - \frac{(\zeta-\beta_k)^{\gamma_k-1}}{(z-\beta_k)^{\gamma_k}} - \ldots,$$

we find that the term that has $(\zeta-\beta_k)^{-1}$ in the expansion of $\varphi(\zeta) = f(\zeta)/(\zeta - z)$ is

$$-\left[\frac{A^{(k)}_{-1}}{z-\beta_k} + \frac{A^{(k)}_{-2}}{(z-\beta_k)^2} + \ldots + \frac{A^{(k)}_{-\gamma_k}}{(z-\beta_k)^{\gamma_k}}\right]\frac{1}{\zeta-\beta_k} = -G_k(z)\frac{1}{\zeta-\beta_k}.$$

Whence,

$$\operatorname*{Res}_{\zeta=\beta_k}\varphi(\zeta) = -G_k(z).$$

Thus,

$$\frac{1}{2\pi i}\int_C \frac{f(\zeta)\,d\zeta}{\zeta-z} = f(z) - \sum_0^n G_k(z),$$

or

$$f(z)=\sum_{0}^{n} G_k(z)+\frac{1}{2\pi i}\int_C \frac{f(z)\,dz}{\zeta-z}. \tag{8.10}$$

The last formula can be found in a different way. We note that $\sum_0^n G_k(\zeta)$ is a rational function that vanishes at infinity; all its poles lie in the interior of C. The function $\frac{\sum_0^n G_k(\zeta)}{\zeta-z}$ (z is in the interior of C) has the same properties and at infinity has a zero of at least the second order. For this reason, the residue of this function at infinity is zero and, hence,

$$\frac{1}{2\pi i}\int_C \frac{\sum_0^n G_k(\zeta)}{\zeta-z}\,d\zeta=0,$$

whence

$$\frac{1}{2\pi i}\int_C \frac{f(\zeta)\,d\zeta}{\zeta-z}=\frac{1}{2\pi i}\int_C \frac{f(\zeta)-\sum_0^n G_k(\zeta)}{\zeta-z}\,d\zeta.$$

But $f(\zeta)-\sum_0^n G_k(\zeta)$ is analytic at all points of the interior of C, whence by applying Cauchy's formula to the last integral we have

$$\frac{1}{2\pi i}\int_C \frac{f(\zeta)\,d\zeta}{\zeta-z}=\frac{1}{2\pi i}\int_C \frac{f(\zeta)-\sum_0^n G_k(\zeta)}{\zeta-z}\,d\zeta=f(z)-\sum_0^n G_k(z),$$

which is exactly (8.10).

Suppose that there is a sequence of rectifiable Jordan curves $\{C_m\}$ that do not pass through the poles of $f(z)$ and are such that each curve (C_m) lies in the interior of the successive curve (C_{m+1}) and whose interiors for a sufficiently large m contain any given circle $|z|<R$. An additional condition is that

$$\lim_{m\to\infty}\frac{1}{2\pi i}\int_{C_m}\frac{f(\zeta)\,d\zeta}{\zeta-z}=0. \tag{8.11}$$

Then the number of poles of $f(z)$ in the interior of C_m depends on m, i.e. $n=n_m$, and (8.10) yields

$$f(z)=\lim_{m\to\infty}\sum_0^{n_m} G_k(z), \tag{8.12}$$

i.e. $f(z)$ is the limit point of a sequence of the principal parts of the Laurent expansions for $f(z)$ at the poles that lie inside C_m.

Criterion (8.11) is met, for instance, when

$$\overline{\lim_{m\to\infty}} \int_{C_m} |f(\zeta)|\, ds < \infty. \tag{8.13}$$

Indeed, if we denote the distance from the origin of coordinates to C_m by r_m ($r_m \to \infty$ as $m \to \infty$) and assume that z belongs to the circle $|z| < R$, we find that for $r_m > R$

$$\left| \frac{1}{2\pi i} \int_{C_m} \frac{f(\zeta)\, d\zeta}{\zeta - z} \right| < \frac{1}{2\pi (r_m - R)} \int_{C_m} |f(\zeta)|\, ds \to 0 \text{ as } m \to \infty.$$

This estimate shows that if condition (8.13) is met, the remainder term in (8.10) tends to zero uniformly with respect to a value of z that belongs to an arbitrary circle $|z| < R$. Whence, the sequence (8.12) converges uniformly to $f(z)$ in any circle $|z| < R$.

We can obtain a representation for $f(z)$ similar to (8.12) under much more general conditions. Namely, we assume that instead of (8.13), the sequence $\{C_m\}$ satisfies the following relationship:

$$\overline{\lim_{m\to\infty}} \int_{C_m} \frac{|f(\zeta)|}{|\zeta|^{p+1}}\, ds < \infty, \tag{8.14}$$

where p is a nonnegative integer. If we assume that the lengths l_m of the C_m grow no faster than λr_m, where λ is a constant (this is always the case when the C_m are in similar correspondence with respect to the origin of coordinates), we find that

$$\int_{C_m} \frac{|f(\zeta)|}{|\zeta|^{p+1}}\, ds < \max_{C_m} |f(\zeta)| \frac{l_m}{r_m^{p+1}} < \lambda \frac{\max\limits_{C_m} |f(\zeta)|}{r_m^p}.$$

This implies that the condition (8.14) is met if a simpler condition is:

$$\overline{\lim_{m\to\infty}} \frac{\max\limits_{C_m} |f(\zeta)|}{r_m^p} < \infty, \tag{8.15}$$

which admits an infinite growth for $\max\limits_{C_m} |f(\zeta)|$ but not one that is faster than r_m^p.

With conditions (8.14) (or (8.15)) satisfied, we return to (8.10) and in the integrand substitute

$$\frac{1}{\zeta - z} = \frac{1}{\zeta} \frac{1}{1 - \frac{z}{\zeta}} = \frac{1}{\zeta} + \frac{z}{\zeta^2} + \ldots + \frac{z^p}{\zeta^{p+1}} + \frac{1}{\zeta - z} \frac{z^{p+1}}{\zeta^{p+1}}$$

for $\zeta \to z$. We then have

$$f(z)=\sum_0^n G_k(z)+\sum_0^p \frac{z^j}{2\pi i}\int_C \frac{f(\zeta)}{\zeta^{j+1}}\,d\zeta$$
$$+\frac{1}{2\pi i}\int_C \frac{f(\zeta)}{\zeta-z}\,\frac{z^{p+1}}{\zeta^{p+1}}\,d\zeta. \qquad (8.16)$$

Since the poles of $f(\zeta)/\zeta^{j+1}$ that lie in the interior of C are points $\beta_0, \beta_1, \ldots, \beta_n$, we put

$$\operatorname*{Res}_{z=\beta_k} \frac{f(\zeta)}{\zeta^{j+1}} = A_k^{(j)} \qquad (j=0,\ 1, \ldots, p).$$

We then have

$$\sum_{j=0}^p \frac{z^j}{2\pi i}\int_C \frac{z(\zeta)}{\zeta^{j+1}}\,d\zeta=\sum_{j=0}^p (A_0^{(j)}+\ldots+A_n^{(j)})\,z^j=\sum_{k=0}^n P_k(z),$$

where $P_k(z)$ is a polynomial of a degree not greater than p:

$$P_k(z) = A_k^{(0)} + A_k^{(1)}z + \ldots + A_k^{(p)}z^p. \qquad (8.17)$$

Hence, we can rewrite formula (8.16) in the following way:

$$f(z)=\sum_0^n [G_k(z)+P_k(z)]+\frac{1}{2\pi i}\int_C \frac{f(\zeta)}{\zeta-z}\,\frac{z^{p+1}}{\zeta^{p+1}}\,d\zeta. \qquad (8.18)$$

If we go over from C to C_m and, therefore, from n to n_m, we find that if we employ (8.14), the residual term in (8.18),

$$\frac{1}{2\pi i}\int_{C_m} \frac{f(\zeta)}{\zeta-z}\,\frac{z^{p+1}}{\zeta^{p+1}}\,d\zeta,$$

tends to zero uniformly with respect to points z that belong to any fixed circle $|z| < R$. Indeed, for m so large that $r_m > R$ we have

$$\left|\frac{1}{2\pi i}\int_{C_m} \frac{f(\zeta)}{\zeta-z}\,\frac{z^{p+1}}{\zeta^{p+1}}\,d\zeta\right|$$
$$< \frac{1}{2\pi}\int_{C_m} \frac{|f(\zeta)|}{|\zeta|-|z|}\,\frac{|z|^{p+1}}{|\zeta|^{p+1}}\,ds < \frac{R^{p+1}}{2\pi(r_m-R)}\int_{C_m} \frac{|f(\zeta)|}{|\zeta|^{p+1}}\,ds.$$

But in view of condition (8.14) the integrals $\int_{C_m} \frac{|f(z)|}{|\zeta|^{p+1}}\,ds$ are bounded, i.e.

$$\int_{C_m} \frac{|f(\zeta)|}{|\zeta|^{p+1}}\,ds < M < \infty.$$

Hence

$$\left| \frac{1}{2\pi i} \int\limits_{C_m} \frac{f(\zeta)}{\zeta - z} \frac{z^{p+1}}{\zeta^{p+1}} d\zeta \right| < \frac{MR^{p+1}}{2\pi (r_m - R)} \to 0 \text{ as } m \to \infty,$$

and (8.18) yields the following expansions:

$$f(z) = \lim_{n_m \to \infty} \sum_{0}^{n_m} [G_k(z) + P_k(z)], \tag{8.19}$$

which converges uniformly to $f(z)$ in each circle $|z| < R$. We can write this expansion in the form of a series:

$$f(z) = [G_0(z) + P_0(z)] + \sum_{m=0}^{\infty} \{[G_{n_m+1}(z) + P_{n_m+1}(z)] + \ldots + [G_{n_{m+1}}(z) + P_{n_{m+1}}(z)]\}, \tag{8.20}$$

where we must put n_0 zero.

We note that the first terms in (8.19) or (8.20) become infinite at $\beta_0, \ldots, \beta_{n_m}$, i.e. where $f(z)$ becomes infinite. For this reason the uniform convergence of (8.19) must be understood as the uniform convergence of a series obtained by dropping from the given expansion the terms that have poles in the circle $|z| < R$.

Expansions of type (8.19) (for example, (8.12) or (8.20)) constitute partial-fraction expansions for $f(z)$.

8.5

PARTIAL-FRACTION EXPANSIONS FOR sec z, cot z, csc z, tan z

The above method of expanding functions belongs to Cauchy. We will apply it to some cases of practical importance.

(a) **Expansion for sec z.** As contours C_m we choose the perimeters of squares centered at point $z = 0$ with sides parallel to the coordinate axes and each $2m\pi$ long. On the sides parallel to the imaginary axis, $z = \pm m\pi + iy$ and, hence,

$$|\sec z| = \frac{1}{|\cos(\pm m\pi + iy)|} = \frac{1}{|\cos iy|} = \frac{1}{\cosh y}.$$

On the sides parallel to the real axis, $z = x \pm im\pi$ and, hence, (see (3.36))

$$|\sec z| = \frac{1}{|\cos(x \pm im\pi)|} \leqslant \frac{1}{\sinh m\pi}.$$

Using these two inequalities, we arrive at the following estimate:

$$\int\limits_{C_m} |\sec \zeta|\, ds < 2 \int\limits_{-m\pi}^{m\pi} \frac{dy}{\cosh y} + 4m\pi \frac{1}{\sinh m\pi}.$$

Since $\int\limits_{-\infty}^{+\infty} \frac{dy}{\cosh y}$ converges and $\frac{4m\pi}{\sinh m\pi} \to 0$ as $m \to \infty$, the criterions (8.13) and, therefore, (8.11) are met. We can thus use (8.12) in this case.

Inside C_m the function $\sec z = (\cos z)^{-1}$ has poles of the type $(2j-1)\pi/2$, where $-m+1 \leqslant j \leqslant m$; all poles are simple since the zeros of $\cos z$ are. Obviously,

$$\operatorname*{Res}_{z=(2j-1)\frac{\pi}{2}} \sec z = -\frac{1}{\sin\left(\frac{2j-1}{2}\pi\right)} = (-1)^j,$$

which implies that the principal part in a neighborhood of point $z = (2j-1)\pi/2$ is $G_j(z) = (-1)^j [z-(2j-1)\pi/2]^{-1}$. We note, in addition, that $z = 0$ is not a pole for $\sec z$ and, hence, the corresponding principal part must be assumed to be zero. From (8.12) we find that

$$\sec z = \lim_{m\to\infty} \sum_{j=-m+1}^{m} \frac{(-1)^j}{z-(2j-1)\frac{\pi}{2}}$$

$$= \lim_{m\to\infty} \left[\sum_{j=1}^{m} \frac{(-1)^j}{z-(2j-1)\frac{\pi}{2}} + \sum_{j=-m+1}^{0} \frac{(-1)^j}{z-(2j-1)\frac{\pi}{2}} \right].$$

If in the second sum we change j to $1-k$, then k will vary from 1 to m and we obtain

$$\sum_{j=-m+1}^{0} \frac{(-1)^j}{z-(2j-1)\frac{\pi}{2}} = \sum_{k=1}^{m} \frac{(-1)^{k+1}}{z+(2k-1)\frac{\pi}{2}}.$$

For this reason, if we now change from k to j, we have

$$\sec z = \lim_{m\to\infty} \left[\sum_{j=1}^{m} \frac{(-1)^j}{z-(2j-1)\frac{\pi}{2}} + \sum_{j=1}^{m} \frac{(-1)^{j+1}}{z+(2j-1)\frac{\pi}{2}} \right]$$

$$= \lim_{m\to\infty} \sum_{j=1}^{m} (-1)^j \frac{(2j-1)\pi}{z^2-(2j-1)^2\frac{\pi^2}{4}}.$$

We have arrived at the following expansion for sec z:

$$\sec z=\sum_{j=1}^{\infty}(-1)^j\frac{(2j-1)\pi}{z^2-(2j-1)^2\frac{\pi^2}{4}}. \tag{8.21}$$

The way in which we have obtained this series, which is a particular case of (8.12) if condition (8.13) is met, implies that it is uniformly convergent in every circle $|z|<R$ (if we are speaking of uniform convergence, we must exclude from the series all terms that have poles inside the given circle).

(b) **Expansion for cot z.** As C_m we choose the perimeters of squares centered at point $z=0$ with sides parallel to the coordinate axes and each $(2m+1)\pi$ long. Then on the sides parallel to the imaginary axis,

$$z=\pm\left(m+\frac{1}{2}\right)\pi+iy$$

and, hence,

$$|\cot z|=\left|\frac{\cos\left[\pm\left(m+\frac{1}{2}\right)\pi+iy\right]}{\sin\left[\pm\left(m+\frac{1}{2}\right)\pi+iy\right]}\right|=\left|\frac{\sin(iy)}{\cos(iy)}\right|=\left|\frac{e^y-e^{-y}}{e^y+e^{-y}}\right|<1.$$

On the sides parallel to the real axis,

$$z=x\pm i\left(m+\frac{1}{2}\right)\pi$$

and, hence (see (3.36)),

$$|\cot z|=\left|\frac{\cos\left[x\pm i\left(m+\frac{1}{2}\right)\pi\right]}{\sin\left[x\pm i\left(m+\frac{1}{2}\right)\pi\right]}\right|\leqslant\frac{\cosh\left(m+\frac{1}{2}\right)\pi}{\sinh\left(m+\frac{1}{2}\right)\pi}$$

$$=\frac{1+e^{-(2m+1)\pi}}{1-e^{-(2m+1)\pi}}\leqslant\frac{1+e^{-\pi}}{1-e^{-\pi}}=\frac{e^{\pi}+1}{e^{\pi}-1}.$$

Thus, on the sides of C_m,

$$|\cot z|\leqslant\frac{e^{\pi}+1}{e^{\pi}-1}.$$

For this reason, condition (8.15) and, therefore, condition (8.14) are met at $p=0$, and we can employ (8.19).

Inside C_m the function $\cot z=\cos z/\sin z$ has poles $0, \pm\pi, \ldots$ $\ldots, \pm m\pi$, which are simple since the zeros of $\sin z$ are. Obviously,

$$\operatorname*{Res}_{z=k\pi}\cot z=\frac{\cos k\pi}{\cos k\pi}=1,$$

which implies that the principal part $G_k(z)$ of the expansion for $\cot z$ in a neighborhood of $z = k\pi$ is $(z - k\pi)^{-1}$.

The polynomials $P_k(z)$ (see (8.17)) in this case are polynomials of a degree not higher than $p = 0$:

$$P_k(z) = A_k^{(0)} = \operatorname*{Res}_{\zeta = k\pi} \frac{\cot \zeta}{\zeta}.$$

But $\cot \zeta/\zeta$ is obviously even, which implies that its expansion in a Laurent series at the origin of coordinates contains only even powers of ζ and, hence,

$$\operatorname*{Res}_{\zeta=0} \frac{\cot \zeta}{\zeta} = 0.$$

Moreover, points $\zeta = k\pi$ $(k \neq 0)$ are simple poles of $\cot \zeta/\zeta$. Whence

$$\operatorname*{Res}_{\zeta=k\pi} \frac{\cot \zeta}{\zeta} = \frac{\cos k\pi}{k\pi \cos k\pi} = \frac{1}{k\pi}.$$

Hence

$$P_0(z) = 0, \quad P_k(z) = \frac{1}{k\pi} \quad (k \neq 0),$$

and by (8.19)

$$\begin{aligned} \cot z &= \lim_{m\to\infty} \left[\frac{1}{z} + \sum_{k=1}^{m} \left(\frac{1}{z-k\pi} + \frac{1}{k\pi} \right) + \sum_{k=1}^{m} \left(\frac{1}{z+k\pi} - \frac{1}{k\pi} \right) \right] \\ &= \lim_{m\to\infty} \left[\frac{1}{z} + \sum_{k=1}^{m} \left(\frac{1}{z-k\pi} + \frac{1}{z+k\pi} \right) \right] \\ &= \frac{1}{z} + \sum_{k=1}^{\infty} \left(\frac{1}{z-k\pi} + \frac{1}{z+k\pi} \right). \end{aligned} \tag{8.22}$$

We have arrived at the partial-fraction expansion for $\cot z$. The way in which we have obtained this formula (from the general formula (8.19)) implies that the series (8.22) is uniformly convergent in every circle $|z| < R$ if we exclude all terms with poles in this circle.

We write (8.22) as

$$\cot z - \frac{1}{z} = \sum_{1}^{\infty} \left(\frac{1}{z-k\pi} + \frac{1}{z+k\pi} \right),$$

and integrate the new series termwise along an arbitrary curve L starting at the origin of coordinates and not passing through points $k\pi$ $(k = \pm 1, \pm 2, \ldots)$. We get

$$\int_0^z \left(\cot z - \frac{1}{z} \right) dz = \sum_{1}^{\infty} \operatorname{Ln} \left(\frac{k\pi - z}{k\pi} \frac{k\pi + z}{k\pi} \right) = \sum_{1}^{\infty} \operatorname{Ln} \left(1 - \frac{z^2}{k^2\pi^2} \right),$$

where in the right-hand side we have definite values of logarithms, namely values represented by definite values of integrals

$$\int_L \left(\frac{1}{z-k\pi}+\frac{1}{z+k\pi}\right) dz.$$

The integral in the left-hand side is equal to one of the values of Ln (sin z/z). Hence

$$\operatorname{Ln}\frac{\sin z}{z}=\lim_{n\to\infty}\sum_1^n \operatorname{Ln}\left(1-\frac{z^2}{k^2\pi^2}\right),$$

whereby

$$\frac{\sin z}{z}=\lim_{n\to\infty}\prod_1^n\left(1-\frac{z^2}{k^2\pi^2}\right).$$

This formula is usually rewritten in the form of an infinite product:

$$\sin z = z\prod_1^\infty\left(1-\frac{z^2}{k^2\pi^2}\right). \tag{8.23}$$

This is the infinite-product expansion for sin z.

(c) **Expansions for csc z and tan z.** The expansions for sec z and cot z at once yield the expansions for csc z and tan z. Indeed, since csc z = sec ($\pi/2$ — z), from the above formula,

$$\sec z=\lim_{m\to\infty}\sum_{j=-m+1}^{m}\frac{(-1)^j}{z-(2j-1)\frac{\pi}{2}}$$

we find that

$$\csc z=\lim_{m\to\infty}\sum_{j=-m+1}^{m}\frac{(-1)^j}{\frac{\pi}{2}-z-(2j-1)\frac{\pi}{2}}$$

$$=\lim_{m\to\infty}\sum_{j=-m+1}^{m}\frac{(-1)^j}{-\pi(j-1)-z}=\lim_{m\to\infty}\sum_{j=-m+1}^{m}\frac{(-1)^{j-1}}{\pi(j-1)+z}.$$

Substituting k for j — 1, we have

$$\csc z=\lim_{m\to\infty}\sum_{k=-m}^{m-1}\frac{(-1)^k}{z+k\pi}$$

$$=\lim_{m\to\infty}\left[\frac{1}{z}-\left(\frac{1}{z-\pi}+\frac{1}{z+\pi}\right)+\left(\frac{1}{z-2\pi}+\frac{1}{z+2\pi}\right)-\dots\right.$$

$$\left.\dots+(-1)^{m-1}\left(\frac{1}{z-(m-1)\pi}+\frac{1}{z+(m-1)\pi}\right)+(-1)^m\frac{1}{z-m\pi}\right].$$

Inside the brackets we add a term $(-1)^m (z + m\pi)^{-1}$, whose limit is zero (uniformly with respect to values of z that belong to an arbitrary circle $|z| < R$). This yields

$$\csc z = \lim_{m\to\infty} \left[\frac{1}{z} - \frac{2z}{z^2-\pi^2} + \frac{2z}{z^2-(2\pi)^2} - \ldots + (-1)^m \frac{2z}{z^2-(m\pi)^2} \right]$$

$$= \frac{1}{z} + \sum_{1}^{\infty} (-1)^k \frac{2z}{z^2-(k\pi)^2},$$

which is the sought for result.

Similarly, for $\tan z$ we have $\tan z = \cot(\pi/2 - z)$, whence from the formula for $\cot z$, namely

$$\cot z = \lim_{m\to\infty} \left[\frac{1}{z} + \sum_{k=1}^{m} \left(\frac{1}{z-k\pi} + \frac{1}{z+k\pi} \right) \right],$$

we obtain

$$\tan z = \lim_{m\to\infty} \left[\frac{1}{\frac{\pi}{2}-z} + \sum_{k=1}^{m} \left(\frac{1}{\frac{\pi}{2}-z-k\pi} + \frac{1}{\frac{\pi}{2}-z+k\pi} \right) \right]$$

$$= \lim_{m\to\infty} \left[\left(\frac{1}{\frac{\pi}{2}-z} + \frac{1}{-\frac{\pi}{2}-z} \right) + \left(\frac{1}{\frac{3\pi}{2}-z} + \frac{1}{-\frac{3\pi}{2}-z} \right) + \right.$$

$$\left. \ldots + \left(\frac{1}{(2m-1)\frac{\pi}{2}-z} + \frac{1}{-(2m-1)\frac{\pi}{2}-z} \right) + \frac{1}{(2m+1)\frac{\pi}{2}-z} \right].$$

Dropping the term $[(2m+1)\pi/2 - z]^{-1}$, whose limit is zero, we find that

$$\tan z = \lim_{m\to\infty} \sum_{1}^{m} \frac{2z}{z^2 - \frac{(2k-1)^2}{4}\pi^2} = \sum_{1}^{\infty} \frac{2z}{z^2 - \frac{(2k-1)^2}{4}\pi^2},$$

which is the sought for expansion for $\tan z$.

(d) We can now easily find the Laurent expansions of trigonometric functions at the origin of coordinates by using the above expansions, instead of dividing series (see Sec. 6.15).

Consider $\sec z$. This function is analytic in the circle $|z| < \pi/2$ and, therefore, admits a Taylor expansion in this circle. Equation

(8.21) represents the function as the sum of a series:

$$\sec z = \sum_{1}^{\infty} (-1)^j \frac{(2j-1)\pi}{z^2-(2j-1)^2\frac{\pi^2}{4}}$$

$$= \sum_{1}^{\infty} (-1)^j \left[\frac{1}{z-(2j-1)\frac{\pi}{2}} - \frac{1}{z+(2j-1)\frac{\pi}{2}} \right],$$

which converges uniformly in every circle (for one, inside $|z| < \pi/2$). Whence, using Weierstrass's theorem on uniformly convergent series of analytic functions, we can find the Taylor expansion coefficients for $\sec z$ by adding the corresponding Taylor expansion coefficients for the functions in the brackets in the right-hand side of the last formula (see Sec. 6.12). But

$$(-1)^j \left[\frac{1}{z-(2j-1)\frac{\pi}{2}} - \frac{1}{z+(2j-1)\frac{\pi}{2}} \right]$$

$$= -(-1)^j \left[\sum_{0}^{\infty} \frac{z^k}{\left[(2j-1)\frac{\pi}{2}\right]^{k+1}} + \sum_{0}^{\infty} \frac{(-1)^k z^k}{\left[(2j-1)\frac{\pi}{2}\right]^{k+1}} \right]$$

$$= 2(-1)^{j-1} \sum_{0}^{\infty} \frac{z^{2m}}{\left[(2j-1)\frac{\pi}{2}\right]^{2m+1}} \quad \left(|z| < (2j-1)\frac{\pi}{2} \right).$$

Here the coefficients of terms with odd powers are zero, and the coefficient of z^{2m} $(m = 0, 1, 2, \ldots)$ is

$$\frac{2(-1)^{j-1}}{\left[(2j-1)\frac{\pi}{2}\right]^{2m+1}} = 2\left(\frac{2}{\pi}\right)^{2m+1} \frac{(-1)^{j-1}}{(2j-1)^{2m+1}}.$$

For this reason the Taylor expansion coefficients for $\sec z$ of the terms with odd powers of z are zero (this, in fact, is obvious, since $\sec z$ is an even function) and of the terms with even powers, z^{2m}, are represented by the series

$$2\left(\frac{2}{\pi}\right)^{2m+1} \sum_{1}^{\infty} \frac{(-1)^{j-1}}{(2j-1)^{2m+1}}.$$

Hence

$$\sec z = \sum_{m=0}^{\infty} \left[2\left(\frac{2}{\pi}\right)^{2m+1} \sum_{j=1}^{\infty} \frac{(-1)^{j-1}}{(2j-1)^{2m+1}} \right] z^{2m}.$$

We recall that in Sec. 6.15 the same expansion was obtained in an alternative form:

$$\sec z = \sum_{1}^{\infty} (-1)^m \frac{E_{2m}}{(2m)!} z^{2m},$$

where E_{2m} are Euler numbers ($E_0 = 1, E_2 = -1, E_4 = 5, E_6 = -61, \ldots$). Comparing the coefficients of the two series, we get

$$2\left(\frac{2}{\pi}\right)^{2m+1} \sum_{1}^{\infty} \frac{(-1)^{j-1}}{(2j-1)^{2m+1}} = (-1)^m \frac{E_{2m}}{(2m)!} \quad (m = 0, 1, 2, \ldots);$$

in particular,

$$\sum_{1}^{\infty} \frac{(-1)^{j-1}}{2j-1} = \frac{E_0}{2} \frac{\pi}{2} = \frac{\pi}{4},$$

$$\sum_{1}^{\infty} \frac{(-1)^{j-1}}{(2j-1)^3} = -\frac{E_2}{2\times 2!}\left(\frac{\pi}{2}\right)^3 = \frac{\pi^3}{32},$$

$$\sum_{1}^{\infty} \frac{(-1)^{j-1}}{(2j-1)^5} = \frac{E_4}{2\times 4!}\left(\frac{\pi}{2}\right)^5 = \frac{5\pi^5}{1536}.$$

Let us consider the function $\cot z - 1/z$, which is analytic in the circle $|z| < \pi$. To find the Taylor expansion coefficients for this function, we use (8.22), which yields the following representation in the form of a series:

$$\cot z - \frac{1}{z} = \sum_{j=1}^{\infty} \left(\frac{1}{z-j\pi} + \frac{1}{z+j\pi}\right).$$

For each term in this series the Taylor expansion is

$$\frac{1}{z-j\pi} + \frac{1}{z+j\pi} = -\sum_{k=0}^{\infty} \frac{z^k}{(j\pi)^{k+1}} + \sum_{k=0}^{\infty} (-1)^k \frac{z^k}{(j\pi)^{k+1}} = -2\sum_{m=1}^{\infty} \frac{z^{2m-1}}{(j\pi)^{2m}}$$

$$(|z| < \pi j).$$

For this reason the expansion coefficients of terms with even powers of z for $\cot z - 1/z$ are zero (which is obvious since the function is odd), while the expansion coefficients of terms with odd powers, z^{2m-1}, can be written in the form of the series

$$-2\sum_{j=1}^{\infty} \frac{1}{(j\pi)^{2m}} = -\frac{2}{\pi^{2m}} \sum_{j=1}^{\infty} \frac{1}{j^{2m}} \quad (m = 1, 2, \ldots).$$

Hence

$$\cot z - \frac{1}{z} = \sum_{m=1}^{\infty} \left[-\frac{2}{\pi^{2m}} \sum_{j=1}^{\infty} \frac{1}{j^{2m}} \right] z^{2m-1}.$$

In Sec. 6.15 the same expansion was obtained in an alternative form:

$$\cot z - \frac{1}{z} = \sum_{m=1}^{\infty} (-1)^m \frac{2^{2m} B_{2m}}{(2m)!} z^{2m-1}.$$

After comparing the coefficients in the two expansions we have

$$\frac{2}{\pi^{2m}} \sum_{j=1}^{\infty} \frac{1}{j^{2m}} = (-1)^{m-1} \frac{2^{2m} B_{2m}}{(2m)!}.$$

Since the left-hand side is positive, so must be the right-hand side, i.e. $(-1)^{m-1} B_{2m} > 0$. This implies that the Bernoulli numbers B_{2m} $(m = 1, 2, 3, \ldots)$ must have alternating signs (in Sec. 6.15 we saw that $B_2 = 1/6$, $B_4 = -1/30$, $B_6 = 1/42, \ldots$).

The above result yields the following particular relations:

$$\sum_{j=1}^{\infty} \frac{1}{j^2} = \frac{B_2}{2 \times 2!} (2\pi)^2 = \frac{\pi^2}{6},$$

$$\sum_{j=1}^{\infty} \frac{1}{j^4} = -\frac{B_4}{2 \times 4!} (2\pi)^4 = \frac{\pi^4}{90},$$

$$\sum_{j=1}^{\infty} \frac{1}{j^6} = \frac{B_6}{2 \times 6!} (2\pi)^6 = \frac{\pi^6}{945}.$$

8.6
ELLIPTIC FUNCTIONS

Elliptic functions constitute a broad class of meromorphic functions, which play an important role in mechanics, geometry, and number theory. Among the other meromorphic functions they are distinguished by exceptionally simple properties due to the property known as double periodicity.

Let α and β be two complex numbers whose ratio is not a real number. If $\beta/\alpha = a + ib$, then $\alpha/\beta = (a + ib)^{-1} = a/(a^2 + b^2) - ib/(a^2 + b^2)$; for this reason one and only one of the ratios α/β and β/α has a positive imaginary part. If we use the notations of Weierstrass's theory, we must designate one of the numbers $2\omega_1$ and

the other $2\omega_3$, so that $\operatorname{Im}(\omega_3/\omega_1) > 0$. Geometrically this means that the smallest rotation (in absolute value) from vector $2\omega_1$ to $2\omega_3$ is in the positive sense (counterclockwise). Let us denote the parallelogram with a vertex at the origin of coordinates and the sides equal to $2\omega_1$ and $2\omega_3$ by Π_{00} (Fig. 55). We wish to build a meromorphic function $f(z)$ for which $2\omega_1$ and $2\omega_3$ are its *fundamental periods*, i.e. a function that satisfies the following two conditions:

(1) for every complex z and arbitrary integers m_1 and m_3,

$$f(z + 2m_1\omega_1 + 2m_3\omega_3) = f(z);$$

(2) if ω is a period of $f(z)$ so that for every z

$$f(z + \omega) = f(z),$$

then there are integers n_1 and n_3 such that

$$\omega = 2n_1\omega_1 + 2n_3\omega_3.$$

The property we have just noted is called *double periodicity* of $f(z)$, and doubly periodic meromorphic functions are called *elliptic*.

The origin of the name can be explained by the fact that historically these functions appeared in the problem of inversion of elliptic integrals (C. F. Gauss, N. H. Abel, and K. G. J. Jacobi), which, incidentally, the definition of the length of an arc of an arbitrary ellipse comes down to. A particular case of such inversion will be considered in Sec. 10.9, where we will use geometric reasoning. Here we will deal only with the purely analytic approach to elliptic functions, an approach independent of the notion of elliptic integrals. This approach was developed in the works of J. Liouville and K. T. W. Weierstrass.

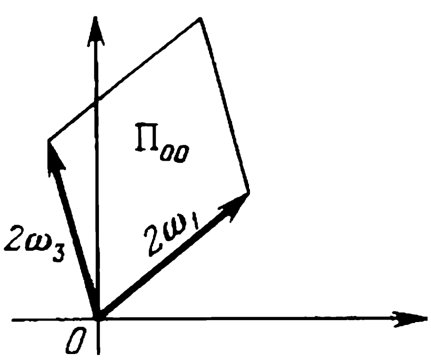


Fig. 55

We start by developing some general properties of elliptic functions with fundamental periods $2\omega_1$ and $2\omega_3$ (for the time being we assume that such functions exist). By translating the *fundamental period parallelogram* we can cover the entire plane by a network of parallelograms congruent to Π_{00} (Fig. 56). Each of these parallelograms is called a *period parallelogram* of $f(z)$, and the set of all vertices coincides with the set of all periods

$$\Omega = 2m_1\omega_1 + 2m_3\omega_3, \qquad m_1, m_3 = 0, \pm 1, \pm 2, \pm 3, \ldots;$$

this set is called a *point lattice*, or a *parallelogram lattice*.

If we want every point of the complex plane to belong to only one period parallelogram, we must agree to ascribe to this point only the two sides of the parallelogram emerging from this point as from the lower left vertex and exclude the end points of the two vectors.

For instance, the period parallelogram $EFGH$ in Fig. 56 has belonging to it only one vertex E (the period $2\omega_1 + 2\omega_3$) and only two sides EF and EG. To each point in the z plane there corresponds one and only one point $z_0 \in \Pi_{00}$ with which z is superposed if we translate the parallelogram that belongs to z to Π_{00}. If $2m_1\omega_1 + 2m_3\omega_3$ is the period belonging to the parallelogram at z (we denote it by $\Pi_{m_1m_3}$), then $z = z_0 + 2m_1\omega_1 + 2m_3\omega_3$, whereby $f(z) = f(z_0)$. In other words, all the values of an elliptic function in a plane are admitted in Π_{00} (and, in general, in any period parallelogram).

The above reasoning implies that if $f(z)$ has not a single pole inside Π_{00}, it does not have a single pole in the entire finite plane, i.e. $f(z)$ is an entire function. But we can easily see that an entire elliptic function is simply a constant. Indeed, if $\max\limits_{z \in \Pi_{00}} |f(z)| = M$, then at every point of the plane

$$|f(z)| \leqslant M,$$

whence, by Liouville's theorem,

$$f(z) \equiv \text{const}.$$

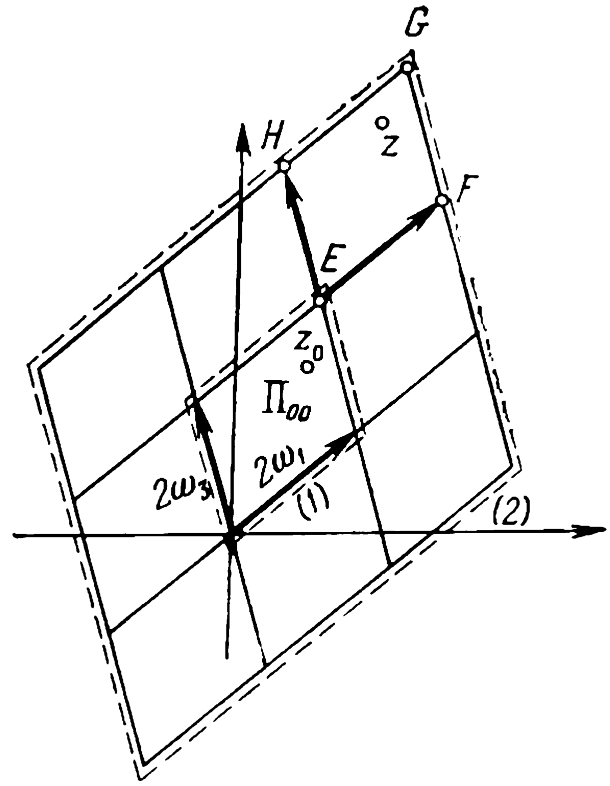


Fig. 56

Since the number of poles of a meromorphic function $f(z)$ in a period parallelogram is finite, it is advisable to speak of the sum of their orders, known as the *order of the elliptic function*, and the sum of residues of $f(z)$ at these poles. It follows then that the order of an elliptic function is at least unity. Now we will prove that the order cannot be less than 2. This statement follows from the fact that the residues of $f(z)$ at poles in any period parallelogram add up to zero. Indeed, this sum is equal to the value of the integral $\frac{1}{2\pi i}\int\limits_{\Gamma} f(z)\,dz$, with Γ the contour of the parallelogram $OABC$ (Fig. 57). But if we construct the Riemann sums for $\int\limits_{OA} f(z)\,dz$ and $\int\limits_{CB} f(z)\,dz$ in a way such that the subdivision points z_k of side OA correspond to the subdivision points $z_k + 2\omega_3$ of CB, we see at once that the integrals are equal and hence $\int\limits_{OA} + \int\limits_{BC} = 0$. The same is true for the other two sides. We note that when there are poles on the contour of $OABC$, to calculate the sum of the residues we must shift the integration

contour somewhat so that all poles belonging to Π_{00} are strictly inside this contour. (In Fig. 58, the closure of Π_{00} contains only eight poles designated by asterisks, but only three of them lie inside the contour of $O'A'B'C'$ and, by the above agreement, belong to one period parallelogram.) Now it is clear why the existence of only one (simple) pole in Π_{00} would contradict what we have just established.

The order of an elliptic function in some respect is similar to the degree of a polynomial. We wish to show that *if r is the order of an elliptic function $f(z)$, for each complex number A the number ν of roots of the equation $f(z) - A = 0$ in any period parallelogram is*

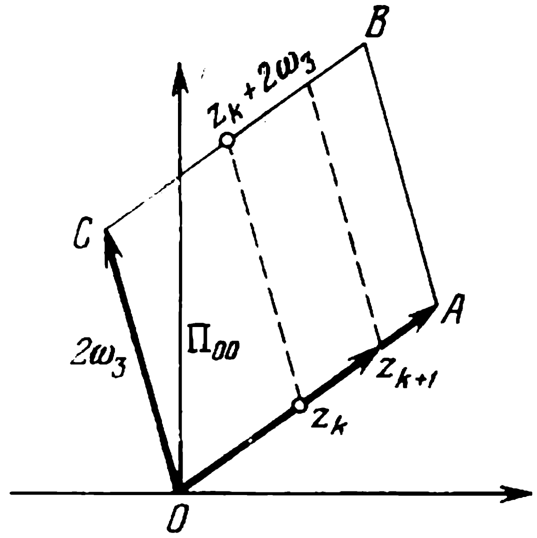


Fig. 57

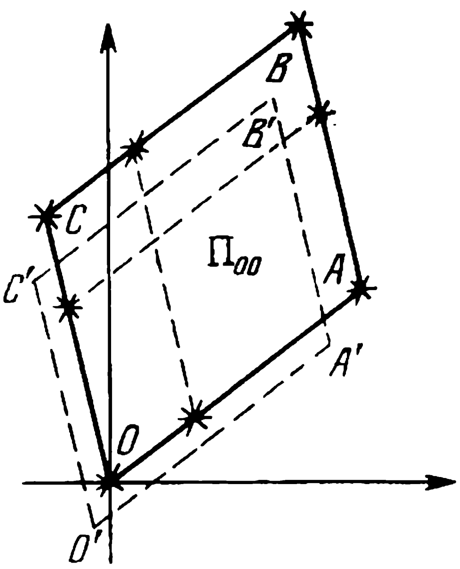


Fig. 58

the same and equal to r. By using the results of Sec. 8.2, it is sufficient to show that $\frac{1}{2\pi i}\int\limits_{OABC}\frac{f'(z)}{f(z)-A}dz = 0$, since this integral is equal to the difference between ν and r.

But $\varphi(z)=\frac{f'(z)}{f(z)-A}$ is, first, meromorphic because both $f'(z)$ and $f(z)-A$ are and, second, has periods $2\omega_1$ and $2\omega_3$ because both $f'(z)$ and $f(z)-A$ have (from $f(z+\omega)=f(z)$ it follows that $f'(z+\omega)=f'(z)$). Reasoning as in the case with $f(z)$, we see that $\frac{1}{2\pi i}\int\limits_{OABC}\varphi(z)\,dz=0.$

Let us now construct Weierstrass's elliptic function of the third order. From this after integration we obtain Weierstrass's elliptic function of the lowest possible order, i.e. order 2.

If we return to Fig. 56, we can see that the boundary of Π_{00} contains four periods; there are eight parallelograms adjoining Π_{00}, and on the boundaries of these there are $12 = 4 + 8$ periods besides the above-mentioned. It is easily seen that to the parallelograms

which adjoin the ones that in Fig. 56 are enveloped by the outermost dotted line there will be added $12+8$ periods. We can continue this process along the same lines (see Fig. 56), so that the kth contour will embrace $(2k-1)^2$ period parallelograms. Then the kth contour will hold $8k-4$ periods of the function (use the method of mathematical induction). This implies that the series $\sum' \frac{1}{\Omega^3}$, where summation is carried out with respect to all nonzero periods $\Omega = 2m_1\omega_1 + 2m_3\omega_3$, is absolutely convergent (*Eisenstein's lemma*). Indeed, let us take the moduli of the periods and consider the sum of a finite number of the series terms corresponding to periods lying on the kth contour. If the distance from point 0 to the first contour is d, the distance to the kth contour will be kd. Whence

$$\frac{1}{|\Omega|^3} \leqslant \frac{1}{k^3 d^3},$$

and the entire sum cannot exceed

$$\frac{8k-4}{k^3d^3} < \frac{8}{d^3}\,\frac{1}{k^2}.$$

This proves the proposition $\left(\text{with an estimate } \sum' \frac{1}{|\Omega|^3} < \frac{8}{d^3} \times \times \sum_{1}^{\infty} \frac{1}{k^2} = \frac{4\pi^2}{3d^3}\right)$.

Let us now consider the series

$$\sum \frac{1}{(z-\Omega)^3},$$

where summation is carried over all the periods of the lattice. We wish to show that in each circle $|z| \leqslant R$ the series converges absolutely and uniformly (provided that we exclude from it a finite number of terms with poles inside or on the boundary of the circle). Indeed, each term for which $|\Omega| > 2R$ admits the following estimate at $|z| \leqslant R$:

$$\frac{1}{|z-\Omega|^3} = \frac{1}{\left|1-\frac{z}{\Omega}\right|^3}\,\frac{1}{|\Omega|^3} < \frac{8}{|\Omega|^3}.$$

From this and Eisenstein's lemma follows our proposition. Consider

$$\varphi(z) = \sum \frac{1}{(z-\Omega)^3}.$$

The above-proved proposition implies that this is a meromorphic function. All its poles have the same order (equal to 3) and each lies at the points of the period lattice.

Let us see whether $2\omega_1$ and $2\omega_3$ are the fundamental periods of this function. We take

$$\varphi(z+2\omega_1) = \sum \frac{1}{[z-(\Omega-2\omega_1)]^3}.$$

The last expansion differs from the one for $\varphi(z)$ in only one respect: each period Ω is translated by the vector $2\omega_1$. Since in the process the period lattice transforms into itself, the sets of terms in the expansions for $\varphi(z)$ and $\varphi(z+2\omega_1)$ coincide. From the fact that the two series are absolutely convergent it follows that their sums are also the same, i.e. $\varphi(z+2\omega_1) = \varphi(z)$. We can prove that $\varphi(z+2\omega_3) = \varphi(z)$ in a similar manner.

But we still have to check whether $\varphi(z)$ has any other period aside from those represented by the given point lattice. Indeed, suppose that

$$\varphi(z+\omega) = \varphi(z)$$

for any z. If $z \to 0$, then $\varphi(z) \to \infty$, whence $\varphi(z+\omega) \to \infty$, i.e. ω is one of the poles of $\varphi(z)$ and, hence, $\omega = 2m_1\omega_1 + 2m_3\omega_3$, with m_1 and m_3 integers.

Thus, $\varphi(z)$ is an elliptic function with fundamental periods $2\omega_1$ and $2\omega_3$. Its order is 3 because to Π_{00} there belongs only one pole of order 3 ($z = 0$). The definition of this function implies that it is odd:

$$\varphi(-z) = \sum \frac{1}{(-z-\Omega)^3} = -\sum \frac{1}{[z-(-\Omega)]^3} = -\sum \frac{1}{(z-\Omega)^3} = -\varphi(z)$$

(we have used the fact that a symmetry transformation with respect to the origin of coordinates maps the point lattice into itself). This also means that three and only three zeros of $\varphi(z)$ belong to Π_{00}.

Now we take the difference $\varphi(z) - 1/z^3$ (point $z = 0$ is a regular point for this function) and integrate it from 0 to z along any path that does not pass through the poles of $\varphi(z)$. We find that

$$\psi(z) = \int_0^z \left[\varphi(z) - \frac{1}{z^3}\right] dz = -\frac{1}{2} \sum{}' \left[\frac{1}{(z-\Omega)^2} - \frac{1}{\Omega^2}\right],$$

where the summation is over all nonzero Ω's.

This new function is meromorphic, and each point of our lattice except $z = 0$ is a pole of order 2. We introduce *Weierstrass's $\wp$ function*

$$\wp(z) = \frac{1}{z^2} + \sum{}' \left[\frac{1}{(z-\Omega)^2} - \frac{1}{\Omega^2}\right].$$

It is obvious that

$$\wp'(z) = \left[-2\int_0^z \left[\varphi(z) - \frac{1}{z^3}\right] dz\right]' - \frac{2}{z^3} = -2\varphi(z)$$

is an elliptic function. Whence

$$\wp'(z+2\omega_j)-\wp'(z)=0 \quad (j=1,\ 3),$$

which implies that

$$\wp(z+2\omega_j)-\wp(z)=2\eta_j \quad (j=1,\ 3),$$

with $2\eta_1$ and $2\eta_3$ constant.

The above definition specifies that the function is even:

$$\wp(-z)=\frac{1}{z^2}+\sum{}'\left[\frac{1}{(-z-\Omega)^2}-\frac{1}{\Omega^2}\right]$$

$$=\frac{1}{z^2}+\sum{}'\left\{\frac{1}{[z-(-\Omega)]^2}-\frac{1}{(-\Omega)^2}\right\}=\wp(z).$$

Here we have relied on the absolute convergence of the series, which follows from Eisenstein's lemma: if $|z|\leqslant R$ and $|\Omega|>2R$, then

$$\left|\frac{1}{(z-\Omega)^2}-\frac{1}{\Omega^2}\right|=\frac{1}{|\Omega|^2}\left|\frac{1}{\left(1-\frac{z}{\Omega}\right)^2}-1\right|$$

$$=\frac{1}{|\Omega|^2}\frac{\left|2\frac{z}{\Omega}-\frac{z^2}{\Omega^2}\right|}{\left|1-\frac{z}{\Omega}\right|^2}<\frac{R}{|\Omega|^3}\frac{\frac{5}{2}}{\frac{1}{4}}=\frac{10R}{|\Omega|^3}.$$

If in $\wp(z+2\omega_j)-\wp(z)=2\eta_j$ $(j=1,\ 3)$ we put $z=-\omega_j$, we get $\wp(\omega_j)-\wp(-\omega_j)=2\eta_j$, which means that

$$2\eta_1=2\eta_3=0.$$

Thus, $2\omega_1$ and $2\omega_3$ are periods of $\wp(z)$. If ω is a period of this function, i.e. $\wp(z+\omega)=\wp(z)$, then we find that $\wp(z)\to\infty$ as $z\to 0$, which implies that ω is a pole of $\wp(z)$, i.e. ω is one of the lattice points. Therefore, $\wp(z)$ is an elliptic function and $2\omega_1$ and $2\omega_3$ are its fundamental periods. Moreover, since in Π_{00} the only pole that belongs to this function is of second order (at $z=0$), the order of $\wp(z)$ is 2.

This is one of the two main Weierstrass's elliptic functions. The other (of order 3) is its derivative

$$\wp'(z)=-2\sum\frac{1}{(z-\Omega)^3}=-2\varphi(z).$$

We put $\omega_2=\omega_1+\omega_3$; then ω_2 is the center of Π_{00}. From

$$\wp(\omega_j-z)=\wp(-\omega_j-z)=\wp(\omega_j+z) \quad (j=1,\ 2,\ 3)$$

it follows that at points pairwise symmetric with respect to ω_j, the function $\wp(z)$ admits the same values. But this directly implies that

$$\wp'(\omega_j)=\lim_{z\to 0}\frac{\wp(\omega_j+z)-\wp(\omega_j-z)}{2z}=0 \quad (j=1,\ 2,\ 3),$$

i.e. ω_1, ω_2, and ω_3 are the three zeros of $\wp'(z) = -2\varphi(z)$ which we have mentioned before. We put

$$\wp(\omega_j) = e_j, \quad (j = 1, 2, 3).$$

We note that e_1, e_2, and e_3 are pairwise distinct, because if we assume that, say, $e_1 = e_2 = e$, we find that $\wp(z) = e$ has two zeros of order 2 in $\Pi_{00}: z = \omega_1$ and $z = \omega_2$, i.e. altogether 4 zeros in Π_{00}, which for a second-order function is impossible.

Consider the function

$$F(z) = 4[\wp(z) - e_1][\wp(z) - e_2][\wp(z) - e_3].$$

This is obviously an elliptic function of order 6 because in Π_{00} it has a single pole of order 6 at $z = 0$, with the principal part being $4/z^6$. Besides, at each of the points ω_1, ω_2, and ω_3 it has one zero of order 2 (the order of these zeros is not greater than 2 because $\wp(z)$ is of order 2).

Let us compare $F(z)$ with $[\wp'(z)]^2$. The latter is also an elliptic function and its order is also 6 because at $z = 0$ it has a pole of order 6; we also note that its principal part is $4/z^6$, as it should be from the fact that $\wp'(z) = -2\sum \frac{1}{(z-\Omega)^3}$. Moreover, $[\wp'(z)]^2$ has three zeros at points ω_1, ω_2, and ω_3, each of order 2. Whence the function $[\wp'(z)]^2/F(z)$, which obviously is meromorphic and whose periods are $2\omega_1$ and $2\omega_3$, has no poles in Π_{00}. Indeed, the poles and zeros of the numerator and denominator coincide, and so do their orders. Therefore, the ratio is a constant. From the fact that

$$\frac{[\wp'(z)]^2}{F(z)} = \frac{\frac{4}{z^6} + \dots}{\frac{4}{z^6} + \dots}$$

at point $z = 0$, where the omitted terms are the regular parts of the Laurent expansions of the numerator and denominator, it follows that this constant is unity. We have thus arrived at the identity

$$[\wp'(z)]^2 = 4[\wp(z) - e_1][\wp(z) - e_2][\wp(z) - e_3].$$

We have proved that *Weierstrass's function* $w = \wp(z)$ *satisfies a first-order differential equation*:

$$\left(\frac{dw}{dz}\right)^2 = 4(w - e_1)(w - e_2)(w - e_3) \quad (e_1 \neq e_2 \neq e_3).$$

If we write the equation as

$$dz = \frac{dw}{\sqrt{4(w - e_1)(w - e_2)(w - e_3)}}$$

and integrate from 0 to z $(\wp(0) = \infty)$, we obtain

$$z = \int_{\infty}^{w} \frac{dw}{\sqrt{4(w-e_1)(w-e_2)(w-e_3)}}.$$

This implies that $w = \wp(z)$ *is the inverse of an elliptic integral of the first kind.*

We note without proof that every elliptic function $f(z)$ with fundamental periods $2\omega_1$ and $2\omega_3$ can always be represented as a rational function of Weierstrass's functions $\wp(z)$ and $\wp'(z)$ with the same periods. More specifically, $f(z)$ can be written in the form

$$f(z) = R[\wp(z), \wp'(z)].$$

It is obvious that such a function is meromorphic and doubly periodic (for one, a constant). The converse statement requires a proof, however.

9

ANALYTIC CONTINUATION. THE RIEMANN SURFACE. SINGULAR POINTS

9.1

THE CONCEPT OF ANALYTIC CONTINUATION

The object of analytic continuation of a function $f(z)$ defined on a set E is to extend the definition of this function to a broader domain $D \supset E$, an extension in which $f(z)$ is analytic in D, too. The simplest example of analytic continuation is the transition from the functions e^x, $\sin x$, $\cos x$ of a real variable (i.e. functions defined only on the real axis E) to the functions e^z, $\sin z$, $\cos z$ of a complex variable, which are analytic in the entire complex plane and coincide in E with the corresponding functions of the real variable. This transition can be achieved by substituting in the power series

$$e^x = \sum_0^\infty \frac{x^n}{n!}, \quad \sin x = \sum_1^\infty (-1)^{n-1} \frac{x^{2n-1}}{(2n-1)!}, \quad \cos x = \sum_0^\infty (-1)^n \frac{x^{2n}}{(2n)!}$$

the complex variable z for the real variable x and noting that the series remain convergent in the process.

Another example is the power series $\sum\limits_0^\infty z^n$, which converges in the unit circle E: $|z| < 1$. The analytic function defined by this series in the unit circle is $f(z) = \sum\limits_0^\infty z^n = 1/(1-z)$. Although outside the unit circle the series is divergent, we can continue $f(z)$ analytically to a domain D that is the entire complex plane without point $z = 1$; it then suffices to put $f(z) = 1/(1-z)$ in D.

The next (and last) example will be the Mittag-Leffler function $E(z)$. It is defined from the start in a domain G_0 that is a complement of the closure of a half-strip D with a boundary L_0 (Fig. 59) by means of the integral

$$E(z) = \frac{1}{2\pi i} \int\limits_{L_0} \frac{e^{e^\zeta}}{\zeta - z} d\zeta.$$

Let us see whether $E(z)$ is analytic in G_0. To this end it suffices to represent $E(z)$ as the limit of a uniformly converging sequence of analytic functions in every finite closed circle $\overline{K}$ that lies in G_0. We take n to be an arbitrarily large number. Then if we separate L_0 into two parts, $L_1^{(n)}$ ($\operatorname{Re}\zeta \leqslant n$) and $L_2^{(n)}$ ($\operatorname{Re}\zeta > n$), we have

$$E(z) = \frac{1}{2\pi i}\int\limits_{L_1^{(n)}} + \frac{1}{2\pi i}\int\limits_{L_2^{(n)}} .$$

But the first term on the right-hand side is a function analytic outside $L_1^{(n)}$ (an integral of the Cauchy type). As to the second, by

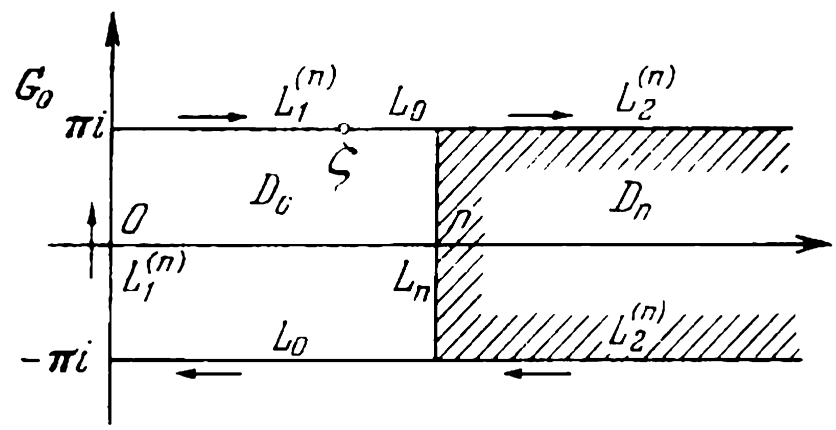


Fig. 59

denoting the distance from $\overline{K}$ to L_0 by d we have

$$\left|\frac{1}{2\pi i}\int\limits_{L_2^{(n)}} \frac{e^{e^{\zeta}}\,d\zeta}{\zeta - z}\right| \leqslant \frac{1}{2\pi d}\int\limits_{L_2^{(n)}} |e^{e^{\zeta}}|\,|d\zeta|$$

$$= \frac{1}{2\pi d}\left(\int\limits_n^{\infty} |e^{e^{\xi+\pi i}}|\,d\xi + \int\limits_n^{\infty} |e^{e^{\xi-\pi i}}|\,d\xi\right) = \frac{1}{\pi d}\int\limits_n^{\infty} e^{-e^{\xi}}\,d\xi \to \infty, \quad n \to \infty.$$

From these facts follows the analyticity of $E(z)$ in G_0.

Let us see whether $E(z)$ can be continued analytically to the entire plane, which would imply that $E(z)$ is an entire function. It suffices to show that for all values of n it is analytically continuable to the domain G_n, the complement to the closure of the half-strip D_n (Fig. 59).

Indeed, if $z \in G_0$, then $\frac{1}{2\pi i}\int\limits_{L_0} \frac{e^{e^{\zeta}}}{\zeta - z}\,d\zeta$ can be represented as a sum of two integrals, one extended to the entire boundary L_n of D_n and the other to the entire boundary of the rectangle $0 < x < n$ and $-\pi < y < \pi$. But according to Cauchy's integral theorem the second integral is zero (we note that $z \in G_0$ lies outside the rectangle).

Whence

$$E(z) = \frac{1}{2\pi i} \int\limits_{L_n} \frac{e^{e^\zeta}\, d\zeta}{\zeta - z}, \quad z \in G_0.$$

Since the right-hand side contains a function analytic in the entire domain G_n, we obtain a continuation of $E(z)$ to G_n and, because n is arbitrarily large, to the entire plane. Therefore, $E(z)$ is an entire function.

As a useful exercise the reader is advised to prove that $E(z)$ tends to 0 uniformly as $z \to \infty$ if an angle with an arbitrary opening span and situated symmetrically to the positive semiaxis is excluded from the plane. On the positive semiaxis $E(x) \to +\infty$ as fast as e^{e^x}. This implies that the entire function $E(z) \exp[-E(z)] \to 0$ as $z \to \infty$ on each ray emerging from point $z = 0$.

Let us now turn from these special cases to the general problem. We suppose at first that the possibility of continuation of $f(z)$ to a domain D as a single-valued and analytic function is established and we only need to know how to calculate its values in this domain. The method for this process coincides in essence with the proof of the uniqueness theorem (Sec. 6.8). We will consider it here in greater detail. Suppose that $E \subset D$ is a set of points having at least one limit point z_0 in D.

We start by finding the expansion coefficients of the series

$$f(z) = c_0^{(0)} + c_1^{(0)}(z - z_0) + \ldots + c_n^{(0)}(z - z_0)^n + \ldots, \tag{9.1}$$

defining $f(z)$ in a neighborhood of z_0. We know already that the series converges in the circle K_0: $|z - z_0| < r_0$, where r_0 is the distance from z_0 to the boundary Δ of D (but the series may converge in a larger circle). Let $\{z_k\}$ be a sequence of points from E distinct from z_0 and from each other, a sequence that converges to z_0; we assume the values $f(z_k)$ $(k = 1, 2, 3, \ldots)$ to be known.

For $c_0^{(0)} = f(z_0)$ we obviously have

$$c_0^{(0)} = \lim_{z_k \to z_0} f(z_k).$$

Let us assume that we have already calculated $c_0^{(0)}, c_1^{(0)}, \ldots$ $\ldots, c_{n-1}^{(0)}$. Then from (9.1) we have

$$c_n^{(0)} = \lim_{z_k \to z_0} \frac{f(z_k) - c_0^{(0)} - c_1^{(0)}(z_k - z_0) - \ldots - c_{n-1}^{(0)}(z_k - z_0)^{n-1}}{(z_k - z_0)^n}.$$

In this way we can calculate all the expansion coefficients of (9.1) one after another. Let us suppose that z' is an arbitrary point that belongs to a domain lying outside the circle K_0. We connect z' and z_0 by a broken line $L \subset D$ and denote the distance between L and Δ by $\delta > 0$. Next we divide L into arcs whose lengths are smaller

than δ and order the subdivision points from z_0 to z' in the following way: $z_0, z_1, z_2, \ldots, z_{m-1}, z_m = z'$. If r_j is the distance from z_j to Δ in the circle K_j: $|z - z_j| < r_j$ the function $f(z)$ can be written thus:

$$f(z) = c_0^{(j)} + c_1^{(j)}(z - z_j) + \ldots + c_n^{(j)}(z - z_j)^n + \ldots \quad (9.2)$$

For $j = 0$ the coefficients are known. Let us suppose that they have been calculated for (9.2) for a $j < m$. Noting that the distance between the centers z_j and z_{j+1} of the circles K_j and K_{j+1} is smaller than the length of the arc with end points z_j and z_{j+1} and, hence, smaller than $\delta \leqslant r_j$, we conclude that z_{j+1} belongs to K_j. For this reason to calculate the coefficients $c_n^{(j+1)} = f^{(n)}(z_{j+1})/n!$ we can use the expansion of $f(z)$ in K_j. We have

$$c_n^{(j+1)} = \frac{f^{(n)}(z_{j+1})}{n!} = c_n^{(j)} + \frac{(n+1)}{1!} c_{n+1}^{(j)}(z_{j+1} - z_j) + \frac{(n+2)(n+1)}{2!} c_{n+2}^{(j)}(z_{j+1} - z_j)^2 + \ldots .$$

In this manner we can calculate the coefficients of each subsequent series of the type (9.2) in terms of those of the previous series. Hence, we are now able to calculate the expansion coefficients of the series with $j = m$, specifically $c_0^{(m)} = f(z_m) = f(z')$. We have thus solved our problem completely. The reader can see that the line of reasoning employed in proving the uniqueness theorem was used here without alterations. The major role was played by the sequence of circles K_j, in each of which there was defined a single-valued analytic function $f_j(z)$ (the sum of the power series (9.2)), and each subsequent circle K_{j+1} had a common part with the previous circle K_j, in which common part the values of $f_{j+1}(z)$ and $f_j(z)$ coincided. The next section is devoted to the generalization of this idea, and arbitrary convex domains will be used instead of circles.

9.2

DIRECT ANALYTIC CONTINUATION

We recall that a domain is said to be *convex* if a straight line connecting any pair of points in it lies entirely inside the domain. A circle, triangle, rectangle, half-plane, angle (with an opening span no greater than π), and a strip between two parallel lines are all examples of convex domains. If two convex domains have a nonempty intersection (i.e. at least one common point), the intersection is a convex domain, too (prove this).

Let us take a convex domain G and a function $f(z)$ analytic and single-valued in G. The combination of the function and domain is called the *analytic function element* (or simply *element*); we will

denote an element by $\{G, f(z)\}$*. Two elements $\{G_1, f_1(z)\}$ and $\{G_2, f_2(z)\}$ are considered identical if and only if the domains G_1 and G_2 coincide and $f_1(z) = f_2(z)$ at all points of G_1 (and G_2). If two elements $\{G_1, f_1(z)\}$ and $\{G_2, f_2(z)\}$ are such that (1) the intersection of G_1 and G_2 is nonempty and (2) in the region common to G_1 and G_2 the values of $f_1(z)$ and $f_2(z)$ coincide, we say that they are *direct analytic continuations* of each other.

The elements $\{G_1, f_1(z)\}$, $\{G_2, f_2(z)\}$, . . ., $\{G_n, f_n(z)\}$ constitute a *chain* of (direct) analytic continuations if each subsequent element $\{G_{j+1}, f_{j+1}(z)\}$ is the direct continuation of the previous element. The chain connects the initial element $\{G_1, f_1(z)\}$ with the final element $\{G_n, f_n(z)\}$; the same chain traversed in the opposite direction connects $\{G_n, f_n(z)\}$ with $\{G_1, f_1(z)\}$. Two elements $\{G, f(z)\}$ and $\{D, \varphi(z)\}$ are *analytic continuations* of each other if there exists a chain of elements connecting the two.

Examples. (a) Let us return to Sec. 9.1. Obviously, for each value of j ($j = 0, 1, 2, \ldots, n$), the sum of (9.2) together with the respective circle $K_j: |z - z_j| < r_j$ comprises an analytic function element, which we denote by E_j. Two adjacent elements E_j and E_{j+1} are analytic continuations of each other; the elements $E_0, E_1, \ldots$ . . ., E_n form a chain of analytic continuations connecting E_0 and E_n (in general E_j and E_k. $0 \leqslant j \leqslant n$, $0 \leqslant k \leqslant n$, are analytic continuations of each other). Finally, the process by which we solved the problem of Sec. 9.1 can be called an *analytic continuation*.

From the reasoning of Sec. 9.1 it follows that if for each point $\zeta \in D$ we build a power series

$$f(\zeta) + f'(\zeta)(z-\zeta) + \ldots + \frac{f^{(n)}(\zeta)}{n!}(z-\zeta)^n + \ldots$$

that converges to $f(z)$ in the circle $K(\zeta)$: $|z - \zeta| < \rho$, where ρ is the distance from point ζ to Δ, we arrive at an infinitude of elements $\{K(\zeta), f(z)\}$, each of which is an analytic continuation of another; the totality of these elements may serve as a definition of $f(z)$ in D.

(b) Let G_j be a half-plane defined by the following inequalities for the polar angle φ: $j\pi/2 < \varphi < (j+2)\pi/2$ ($j = 0, \pm 1, \pm 2, \ldots$); obviously $G_{j+4n} = G_j$ (n an integer). We put $f_j(z) = \ln|z| + i\varphi$, where φ satisfies the same inequalities; this yields an element $\{G_j, f_j(z)\}$. It is easy to see that for any integral value of j the elements $\{G_j, f_j(z)\}$ and $\{G_{j+1}, f_{j+1}(z)\}$ are analytic continuations of each other (the common part of the half-planes G_j and G_{j+1} is the coordi-

* Convexity leads to connectedness of the common part of the two domains. If this restriction is lifted, all previous reasoning remains unchanged if we introduce certain modifications into the statements. These modifications are necessary because in the general case the common part of the two domains consists of many regions without pairwise common points.

nate quadrant $(j+1)\pi/2 < \varphi < (j+2)\pi/2$, where f_j and f_{j+1} have the same values). This implies that any elements $\{G_j, f_j\}$ and $\{G_k, f_k\}$ are analytic continuations of each other. It is also obvious that the collection of all the elements $\{G_j, f_j(z)\}$ can serve as a definition of the multiply valued function $\operatorname{Ln} z$.

9.3

CONSTRUCTING AN ANALYTIC FUNCTION FROM ITS ELEMENTS

For the sake of brevity we will call the set M of elements $\{G, f(z)\}$ (finite or infinite) *connected* if every two elements of M, say $\{G_0, f_0(z)\}$ and $\{G^*, f^*(z)\}$, are analytic continuations of each other and if M contains with each pair of elements a chain of elements connecting the two. For instance, the sets of elements in Examples (a) and (b) of Sec. 9.2 are connected. We wish to show that the union $\bigcup G = D$ of all the domains included in the definition of the elements belonging to a connected set M is a domain, too. Indeed, if $z \in D$, then z belongs to one domain G at least, which implies that a neighborhood of z belonging to G also belongs to D. Let z_0 and z^* be two distinct points from D; let z_0 belong to G_0, a domain that has corresponding to it the element $\{G_0, f_0(z)\} \in M$, and z^* to G^*, which has corresponding to it the element $\{G^*, f^*(z)\}$. We will also assume that $\{G_0, f_0(z)\}, \{G_1, f_1(z)\}, \ldots, \{G_n, f_n(z)\}$ ($f^* = f_n$ and $G^* = G_n$) is the chain of elements from M that connects $\{G_0, f_0(z)\}$ with $\{G^*, f^*(z)\}$. In the intersection of G_j and G_{j+1} ($j = 0, 1, \ldots, n-1$) we select one point z_{j+1}. Then we can connect z_0 with z_1 by a straight line $\delta_1 \subset G_0$ (hence $\delta_1 \subset D$), z_1 with z_2 by a straight line $\delta_2 \subset G_1 \subset D, \ldots, z_{n-1}$ with $z_n = z^*$ by a straight line $\delta_n \subset G_{n-1} \subset D$. It is obvious that the straight lines $\delta_1, \delta_2, \ldots, \delta_n$ constitute a broken line in D that connects z_0 with z^*. We have proved that D is a domain. Now we can define via M the analytic function $f(z)$ in D in the following manner ($f(z)$ may be multiply valued). If $z_0 \in D$, we can take one element, say $\{G_0, f_0(z)\}$ for which $z_0 \in G_0$, and assume that $f(z) = f_0(z)$ in a neighborhood of $z_0 \in G_0$ (in particular, $f(z_0) = f_0(z_0)$). We have thus obtained one branch of the analytic function $f(z)$. Since there may be several distinct elements $\{\tilde{G}, \tilde{f}(z)\}$ for which $z_0 \in \tilde{G}$ (even an infinitude of such elements), we can obtain several (even an infinitude) of neighborhoods of z_0 with a corresponding branch of $f(z)$ in each. To fix one branch and with it a certain value $f(z_0)$ we must obviously select not only a point z_0 but a definite element $\{\tilde{G}, \tilde{f}(z)\} \in M$ for which $z_0 \in \tilde{G}$. Then we will obtain a branch of $f(z)$ not only in a circular neighborhood of z_0 but in the entire domain $\tilde{G}$; this branch is defined by the func-

tion $\tilde{f}(z)$. In what follows we will consider each element $\{\tilde{G}, \tilde{f}(z)\}$ as an element of the built function $f(z)$. For the sake of illustration we recall Examples (a) and (b) of Sec. 9.2. In the first example the union of all circles $K(\zeta)$ yields the initial domain D, and the function $f(z)$ defined via its elements $\{K(\zeta), f(\zeta) + f'(\zeta)(z - \zeta) + \ldots\}$ proves to be single-valued and analytic in D. In the second example the union of all half-planes G_j: $j\pi/2 < \varphi < (j + 2)\pi/2$ ($j = 0, \pm 1, \pm 2, \ldots$) represents the entire plane without points $z = 0$ and $z = \infty$. The elements $\{j\pi/2 < \varphi < (j + 2)\pi/2, \ln|z| + i\varphi\}$ define in this domain an infinitely valued analytic function, namely $\mathrm{Ln}\, z$. In each of the four half-planes $j_0\pi/2 < \varphi < (j_0 + 2)\pi/2$ ($j_0 = 0, 1, 2, 3$) (an infinitude of half-planes G_j for which $j = 4n + j_0$, $n = 0, \pm 1, \pm 2, \ldots$, coincide with each such half-plane) the function $\mathrm{Ln}\, z$ has an infinitude of branches

$$f_{4n+j_0}(z) = \ln|z| + i\varphi + i2\pi n.$$

9.4

CONSTRUCTING RIEMANN SURFACES

G. F. B. Riemann suggested the idea of generalizing the notion of a domain so that any multiply valued analytic function $f(z)$ of a complex variable z becomes single-valued if it is considered as a function of a point in an appropriately generalized domain.

Let M be a connected set of elements $\{G, f(z)\}$. When we built the domain $D = \bigcup G$, each point z belonging to the domain G_0 of an element $\{G_0, f_0(z)\}$ and to the domain G^* of an element $\{G^*, f^*(z)\}$ was considered as one point in D and not as two points.

Now we change the described process of uniting the domains $\{G\}$ and consider the point z belonging to the G_0 and G^* of the elements $\{G_0, f_0(z)\}$ and $\{G^*, f^*(z)\}$ as one point if and only if these elements are direct analytic continuations of each other (i.e. when the values of $f_0(z)$ and $f^*(z)$ coincide in the intersection of G_0 and G^*). For pictorial purposes let us assume that for every element $\{G, f(z)\}$ we have made a paper or cloth model of appropriate configuration. We will envisage the process of uniting $\{G\}$ as the gluing together of the pieces along lines whose points are considered identical. In other words, the pieces that represent the G_0 and G^* of the elements $\{G_0, f_0(z)\}$ and $\{G^*, f^*(z)\}$ become glued together only if two conditions are met simultaneously: (1) G_0 and G^* have a nonempty intersection $G_0 \cap G^* = g$, and (2) $f_0(z) = f^*(z)$ at all points of g. If we have done all the gluing of $\{G\}$ possible under these conditions, we arrive, as a result, at a *generalized domain* R (generally a multisheeted surface) situated above D. In some cases, however, as in Example (a) of Sec. 9.2, the construction of R does not differ in any respect

from the construction of D, since each time when the circles $K(\zeta_0)$ and $K(\zeta^*)$ have a nonempty intersection, the sums of the corresponding power series (9.2) are the same; for this reason the generalized domain R is equivalent to D in this case. In Example (b) of Sec. 9.2 the situation is different. Here the domains G_j: $j\pi/2 < \varphi < (j+2)\pi/2$, with all integral values of j such that when divided by four the remainder is the same and equal to j_0, represent one half-plane $j_0\pi/2 < \varphi < (j_0+2)\pi/2$ bounded by the appropriate coordinate axis. We imagine these domains as an infinitude of distinct paper sheets collected into a single pack above the half-plane G_{j_0}. Since j_0 admits four values 0, 1, 2, and 3, there are four such packs altogether. No pair of sheets G_{4k+j_0} and G_{4l+j_0} $(k \neq l)$ from the same pack can become glued together directly because at their intersection (which coincides with the half-plane G_{j_0}) $f_{4k+j_0} = \ln|z| + i(\varphi + 2k\pi)$ and $f_{4l+j_0} = \ln|z| + i(\varphi + 2l\pi)$ $(j_0\pi/2 < \varphi\,(j_0+2)\,\pi/2,\ k \neq l)$. Only sheets with subscripts that differ by unity, G_j and G_{j+1} (they belong to different packs), become glued along the common part, which lies above the coordinate quadrant $(j+1)\pi/2 < \varphi < (j+2)\pi/2$. Indeed, in this common part the values of the functions $f_j(z) = \ln|z| + i\varphi$ $(j\pi/2 < \varphi < (j+2)\pi/2)$ and $f_{j+1}(z) = \ln|z| + i\varphi$ $((j+1)\pi/2 < \varphi < (j+3)\pi/2)$ coincide.

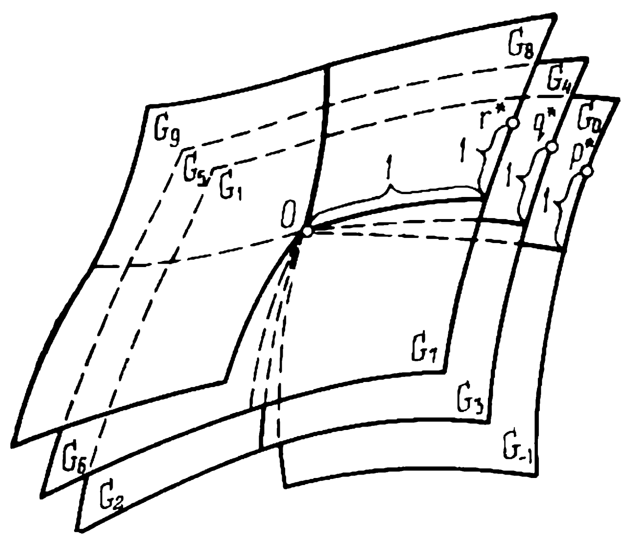


Fig. 60

As a result of all these gluing processes we obtain an infinitely sheeted domain R above the respective domain D (D is the z plane with two points excluded, $z = 0$ and $z = \infty$).

A graphic representation of R is given in Fig. 60, where that part of R is depicted that is obtained by gluing together 11 half-planes $G_{-1}, G_0, G_1, G_2, \ldots, G_9$ (the reader is advised to make a similar model by gluing together 11 rectangular sheets of paper).

It is obvious that to define a point in R it is not enough to assign a complex number z, the affix of this point. For instance, in the last example one affix $1 + i$ has an infinitude of different points on R all lying on the different parts of R as a result of pasting G_{-1} with G_0, G_3 with G_4, G_7 with G_8, ..., and in general G_{4k+3} with G_{4k+4} $(k = 0, \pm 1, \pm 2, \ldots)$ (see Fig. 60). In the general case, to define a point $p \in R$, we must specify, together with the affix z, a definite element $\{G^*, f^*(z)\}$ such that $z \in G^*$. Then only do we have a definite value of the function at this point: $f(p) = f^*(z)$. For instance,

the value of the logarithm at point p^* depicted in Fig. 60 is $\ln \sqrt{2} + i\pi/4$, at point q^* it is $\ln \sqrt{2} + i(\pi/4 + 2\pi)$, and at point r^* it is $\ln \sqrt{2} + i(\pi/4 + 4\pi)$.

Hence, in the generalized domain R built according to the above method, the given connected set of elements, M: $\{G, f(z)\}$, defines a single-valued function of a point. This function is considered analytic in R. To verify that the notion of analyticity is carried over directly to functions defined in R, it suffices to note that defining a point p^* on R means assigning an affix z^* of this point and also isolating a definite element $\{G^*, f^*(z)\}$ from M for which $z^* \in G^*$. If this is done, we have also isolated the part of R above G^* (it is also said that this part of R projects itself onto G^*) in which each point p is completely characterized by its affix $z \in G^*$ (the projection of point p). Thus, the function $f(p)$ of point p in a given part of R can be taken as a function of a complex variable z in G and, therefore, we can apply to it all notions and results of the theory of functions of a complex variable z in G. The generalized domain R described in this section is called the *Riemann surface* of the analytic function $f(z)$ (which, in turn, is defined by a given connected set of elements M: $\{G, f(z)\}$).

9.5
THE RIEMANN-SCHWARZ SYMMETRY PRINCIPLE

At the base of the process of analytic continuation described above we placed the idea of direct analytic continuation of two elements, $\{G_1, f_1(z)\}$ and $\{G_2, f_2(z)\}$, with overlapping domains. The main feature was that with the aid of two functions, $f_1(z)$ and $f_2(z)$, analytic in two domains, G_1 and G_2, we arrive at a function $f(z)$ that is analytic in a greater domain $G = G_1 \cup G_2$ (Fig. 61); here $f(z)$ coincides with $f_j(z)$ in G_j $(j = 1, 2)$. It is worthwhile generalizing the definitions of an element and a direct analytic continuation in the following manner.

We will call the combination of a function $f(z)$ single-valued and analytic in a domain G bounded by a generalized Jordan curve Γ (G is not necessarily a convex domain) and the domain G an *analytic function element*; as before we denote it by $\{G, f(z)\}$. Let $\{G_1, f_1(z)\}$ and $\{G_2, f_2(z)\}$ be two elements, and let the domains G_1 and G_2 have no common points but their boundaries Γ_1 and Γ_2 have a common open arc δ (i.e. without the end points) (Fig. 62*a*). If in the domain $G = G_1 + G_2 + \delta$ (Fig. 62*b*) there exists an analytic function $f(z)$ that coincides with $f_j(z)$ in G_j $(j = 1, 2)$, the elements $\{G_1, f_1(z)\}$ and $\{G_2, f_2(z)\}$ are called as before *direct analytic continuations* of each other. We also say that $f_1(z)$ is *analytically continuable* from G_1 across arc δ to G_2 and that $f_2(z)$ is its analytic continuation (or that $f_2(z)$ is analytically continuable from G_2 across δ

to G_1 and $f_1(z)$ is its analytic continuation). The case we are studying now, i.e. where the domains touch each other, can be reduced to the case where the domains overlap. Indeed, let us connect the end points of δ by arcs γ_1 and γ_2 as shown in Fig. 62*a*, and denote the curves obtained from Γ_1 and Γ_2 by substituting γ_1 and γ_2 for δ by Γ_1' and Γ_2', respectively; we denote the interiors of Γ_1' and Γ_2' by G_1' and G_2'. The domains G_1' and G_2' have a common part bounded by the Jordan curve $\gamma_1 \cup \gamma_2$. Obviously, when $f_1(z)$ is analytically continuable from G_1 to G_2 across δ, we can analytically continue $f_1(z)$ to

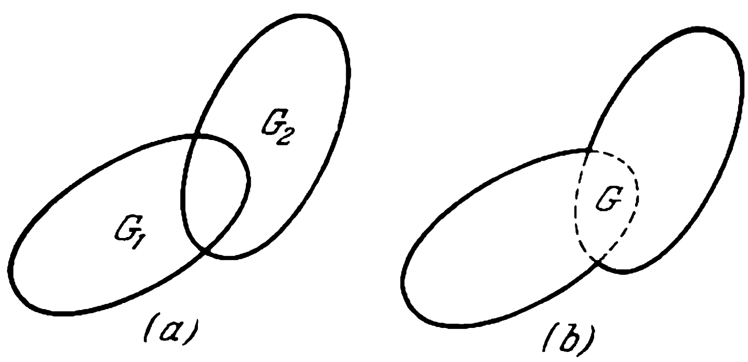


Fig. 61

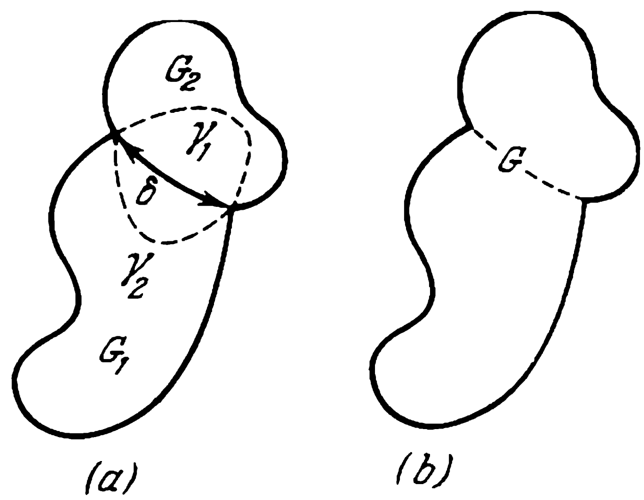


Fig. 62

G_1' and $f_2(z)$ to G_2', and the elements $\{G_1', f_1(z)\}$ and $\{G_2', f_2(z)\}$, whose domains overlap, constitute direct analytic continuations on each other in the former sense (see Sec. 9.2 and the footnote on p. 334). In a similar manner we can prove the converse statement, namely that analytic continuation via elements with overlapping domains can be reduced to analytic continuation via elements with adjoining domains.

On the basis of the concept of direct analytic continuation via elements with adjoining domains we can introduce the notion of a *chain of analytic continuations* (not direct) and then construct a Riemann surface as we did in Secs. 9.2-9.4. In the process of constructing the Riemann surface we must bear in mind that two domains G_0 and G^* of the elements $\{G_0, f_0(z)\}$ and $\{G^*, f^*(z)\}$ are joined (glued) along the common boundary arc δ if and only if $f_0(z)$ is continuable across δ to G^* and $f^*(z)$ is the result of the continuation. The reader is advised to trace how the Riemann surface of $\operatorname{Ln} z$, obtained in Sec. 9.4 by gluing together an infinitude of overlapping copies of four different half-planes (the upper and the lower and the left and the right), can be obtained by gluing together an infinitude of copies of only two different half-planes, say the upper and the lower, which are connected alternately along the positive and negative real semiaxes.

Let us show that $\{G_1, f_1(z)\}$ and $\{G_2, f_2(z)\}$ serve as direct continuations of each other in the specific case where δ is a straight segment and the $f_j(z)$ are continuous on $G_j + \delta$ and at points of δ admit equal values. Indeed, if we put $f(z) = f_j(z)$, $z \in G_j$ $(j = 1, 2)$, and $f(z) = f_1(z) = f_2(z)$, $z \in \delta$, we find that $f(z)$ is continuous in the domain $G = G_1 \cup G_2 \cup \delta$ and analytic in G_j $(j = 1, 2)$. Hence $\int_\gamma f(z)\, dz = 0$, where γ is a triangular contour lying entirely in G_1 or G_2. Then according to Sec. 6.5 $f(z)$ is analytic in G, i.e. the elements $\{G_1, f_1(z)\}$ and $\{G_2, f_2(z)\}$ are analytic continuations of each other. Now it is easy to prove the following important theorem.

The Riemann-Schwarz symmetry principle. *Let G be a domain bounded by a Jordan curve Γ with a straight segment δ. Suppose that*

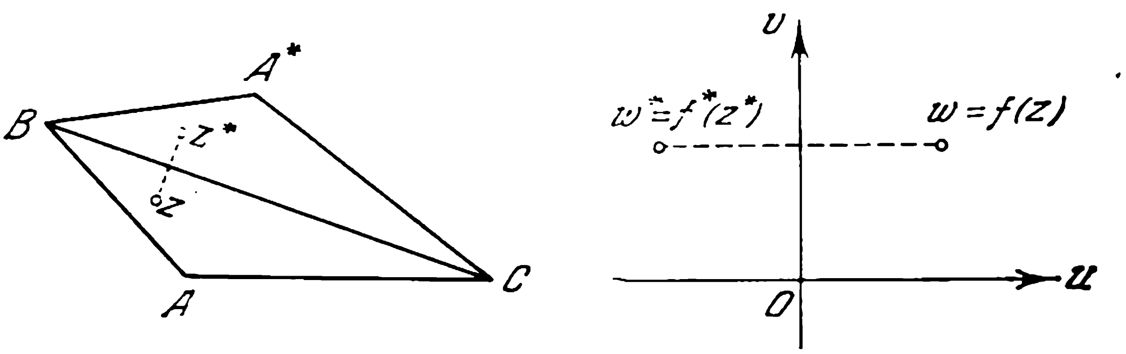


Fig. 63

on $G + \delta$ we have defined a continuous function $f(z)$ analytic in G such that at the points of δ it admits values that lie on a straight line Λ. Let us build a domain G^* symmetric to G with respect to δ (strictly speaking, with respect to a straight line containing δ) and define on $G^* + \delta$ a function $f^*(z)$ by putting $f^*(z) = f(z)$ when $z \in \delta$ and assuming that $f^*(z^*)$ is symmetric to $f(z)$ with respect to Λ if point $z^* \in G^*$ is symmetric to $z \in G$ with respect to δ. Then the elements $\{G, f(z)\}$ and $\{G^*, f^*(z)\}$ are direct analytic continuations of each other.

This theorem provides simple sufficient conditions in which a function $f(z)$ defined in a domain G is continuable to the domain $G + \delta + G^*$, twice as large, so to say, as G; the theorem also shows how to execute this continuation (by the symmetry principle). Before we go over to the proof, let us clarify its meaning by a simple example. Suppose that $w = f(z)$ is continuous in a closed triangle ABC (Fig. 63), analytic inside it, and admits purely imaginary values on BC. The theorem then states that $f(z)$ is analytically continuable to the domain ABA^*C. If the points z^* and z of this domain are symmetric with respect to BC, the values w^* and w of the continued function at these points are symmetric with respect to the imaginary axis (Fig. 63).

Proof of the theorem. We start by subjecting the z and w planes to the linear transformations $z' = az + b$ and $w' = \alpha w + \beta$, choosing a, b, α, and β so that the segment δ becomes a section of the real axis of the z' plane and the straight line Λ becomes the real axis of the w' plane. In the process, of course, both the domains and the functions mentioned in the hypothesis of the theorem will change $\left(\text{for instance, instead of } w = f(z) \text{ we will have } w' = \alpha f\left(\frac{z'-b}{a}\right)+\beta\right)$, but we will keep the same notations for the functions and domains; the same notations will also be kept for the variables z and w. It is essential that these transformations (and also the inverse transformations) map a finite plane into a finite plane, a domain into a domain, and straight lines into straight lines, preserve the symmetric configuration of points in relation to the mapped straight lines, and, finally, preserve the continuity and analyticity of functions. For this reason, when we carry out the proof for the case where the segment δ lies on the real axis and the straight line Λ coincides with the real axis, we will also verify the validity of the theorem in its initial formulation.

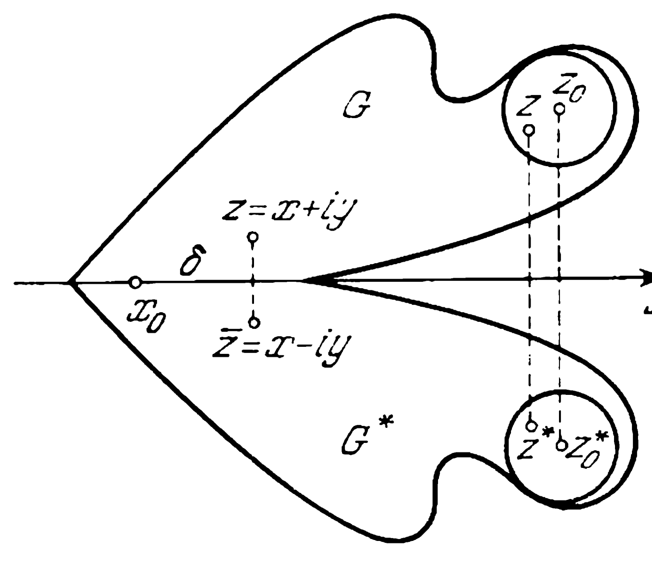


Fig. 64

We note that now at the points of the segment δ we have $z = x$ (i.e. $y = \operatorname{Im} z = 0$) and $w = f(z) = f(x) = u$ (i.e. $v = \operatorname{Im} w = 0$); moreover, the points z and z^* symmetric with respect to δ and also the points w and w^* symmetric with respect to Λ are represented by pairs of mutually conjugate complex numbers. Let us demonstrate that a function $f^*(z^*)$ defined as in the hypothesis of the theorem is analytic in G^* and continuous in $G^* + \delta$. Indeed, if $z_0^* \in G^*$ and if the neighborhood $|z^* - z_0^*| < \rho$ belongs to G^*, then the point $z_0 = \overline{z_0^*} \in G$ and its neighborhood $|z - z_0| < \rho$ belongs to G (Fig. 64).

But in a neighborhood of z_0 the function $f(z)$ can be represented by a power series:

$$f(z) = c_0 + c_1(z - z_0) + \dots + c_n(z - z_0)^n + \dots .$$

For this reason at the symmetric point $z^* = \bar{z}_0$ belonging to the above-mentioned neighborhood of $z_0^* = \bar{z}_0$ we must have, by definition,

$$f^*(z^*) = \overline{f(z)} = \bar{c}_0 + \bar{c}_1(\bar{z} - \bar{z}_0) + \dots + \bar{c}_n(\bar{z} - \bar{z}_0)^n + \dots$$
$$= \bar{c}_0 + \bar{c}_1(z^* - z_0^*) + \dots + \bar{c}_n(z^* - z_0^*)^n + \dots .$$

We see that $f^*(z^*)$ in a neighborhood of point $z_0^* \in G^*$ is represented by a convergent power series and, hence, is an analytic function.

Let us show, finally, that $f^*(z^*)$ is continuous in $G^* + \delta$. Actually we have to verify that the function is continuous at each point $x_0 \in \delta$.

Let us assume for the sake of definiteness that y is greater than zero in a small half-neighborhood of point x_0 in G. By the hypothesis of the theorem,

$$\lim_{z\to x_0,\ z\in G} f(z) = \lim_{\substack{x\to x_0,\ y\to 0\\ (y>0)}} [u(x, y) + iv(x, y)] = f(x_0) = u(x_0, 0),$$

whence

$$\lim_{x\to x_0,\ y\to 0\ (y>0)} [u(x, y) - iv(x, y)] = f(x_0) = u(x_0, 0).$$

Hence, assuming that $z^* = \bar{z}$, where $z = x + iy \in G$, and noting that $z^* \to x_0$ implies that $z = \bar{z}^* \to x_0$, we have

$$\lim_{z^*\to x_0,\ z^*\in G^*} f^*(z^*) = \lim_{z\to x_0,\ z\in G} \overline{f(z)}$$
$$= \lim_{x\to x_0,\ y\to 0\ (y>0)} [u(x, y) - iv(x, y)] = f(x_0) = f^*(x_0).$$

We have thus proved that $f^*(z^*)$ is continuous in $G^* + \delta$. Now the elements $\{G, f(z)\}$ and $\{G^*, f^*(z)\}$ satisfy the conditions stated on p. 339 and, hence, are direct analytic continuations of each other. The proof of the theorem is complete.

The symmetry principle remains valid for the case where instead of a straight segment δ the boundary of G contains an arc γ of a circle c, and the values of $w = f(z)$ at points of γ fall on a circle d instead of a straight line. To reduce this general case to the one considered, we subject the z and w planes to auxiliary linear-fractional mappings (one for each plane) that map c and d into straight lines C and D. In the process G is mapped into a domain B, the arc γ into a straight segment of the boundary of B lying on C, the function $w = f(z)$ into $W = F(Z)$, and points that are pairwise symmetric with respect to c and d into points pairwise symmetric with respect to the straight lines C and D.

Now let us apply the Riemann-Schwarz principle in the form considered to the function $W = F(Z)$. As a result we obtain its analytic continuation to B^*, a domain symmetric to B with respect to C. At points Z and Z^* symmetric with respect to C the values of $F(Z)$ and $F^*(Z^*)$ are symmetric with respect to D. Next we return to the old variables z and w by applying the inverse linear-fractional mappings. Since analyticity and symmetry are preserved in the course of such mappings, the final result is the analytic continuation

$f^*(z)$ of the initial function $f(z)$, where $f^*(z)$ is defined in G^* symmetric to G with respect to the circle c. At points z^* and z symmetric with respect to c the values of $f^*(z^*)$ and $f(z)$ are symmetric with respect to the circle d.

In the above reasoning, however, we must allow for the possibility of poles in the continued function. For instance, if we assume that d is a circle centered at the origin of coordinates and $f(z)$ has zeros in G, then in the process of analytic continuation to G^* each zero becomes a pole at the symmetric point z_0^*, because the values of $f(z_0)$ and $f^*(z_0^*)$ must be symmetric with respect to d.

9.6

SINGULARITIES ON THE BOUNDARY OF THE CONVERGENCE CIRCLE OF A POWER SERIES

Let us clarify the notion of a singularity. We recall (see Sec. 7.3) that when we considered a function $f(z)$ single-valued and analytic in a domain D: $0 < |z - z_0| < R$, we called the point z_0 an isolated boundary point in D, regular or singular depending on whether we could or could not define $f(z)$ at z_0 in a way such that the function became analytic in the entire circle $|z - z_0| < R$ (point z_0 inclusive). It was in this manner we introduced the notion of an isolated singular point, or an isolated singularity.

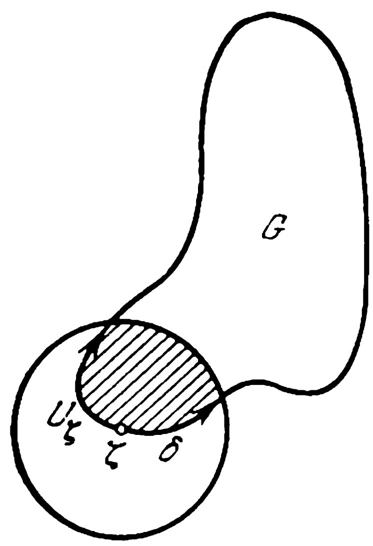


Fig. 65

We will now see how the idea of a regular or singular point can be generalized to include the case of nonisolated boundary points of a domain. Let G be a domain bounded by a Jordan curve Γ and $f(z)$ a single-valued and analytic function in G; then $f(z)$ and G define the element $\{G, f(z)\}$. We will call the point $\zeta \in \Gamma$ a *regular point of element* $\{G, f(z)\}$ (briefly, a *regular point of* $f(z)$) if there is a neighborhood U_ζ: $|z - \zeta| < \rho$ of point ζ and in it a single-valued analytic function $\varphi_\zeta(z)$ coinciding with $f(z)$ in the part common to this neighborhood and G. Obviously, there is an arc δ of Γ that contains point ζ, and across this arc $f(z)$ can be analytically continued (Fig. 65).

If a boundary point ζ is not regular, it is called a *singular point of element* $\{G, f(z)\}$ (briefly, a *singular point of* $f(z)$). In this case there is no single analytic and single-valued function that coincides with $f(z)$ in the part common to a neighborhood of this point and G. In other words, there is no single arc $\delta \subset \Gamma$ containing point ζ across which $f(z)$ can be analytically continued from G.

The definition of a regular point implies that if $\zeta_0 \in \Gamma$ is a regular point, all points of an arc δ containing this point are regular, too. This means, in particular, that if there is at least one regular point, there is an infinitude of such points. In contrast, a function may have only one singular point.

We note that each point $z_0 \in G$ has the characteristic property of a regular point, namely it has a neighborhood K_0 (we can take any neighborhood of z_0 lying in G) and there exists a function that is analytic in K_0 (the function $f(z)$) and that coincides with $f(z)$ at all points common to K_0 and G. For this reason we will also call all points of G regular points of the element $\{G, f(z)\}$.

Example. Let G be the unit circle and $f(z) = 1/(1-z)$. Then for each point ζ lying on the circle Γ: $|z| = 1$ and not unity there exists a neighborhood K: $|z-\zeta| < |1-\zeta|$ and in it an analytic function $\varphi(z) = 1/(1-z)$ coinciding with $f(z)$ at points common to K and G. For this reason each point ζ is regular for $1/(1-z)$. We wish to show that $\zeta_0 = 1$ is a singular point.

If we assume that this is not so, then in a neighborhood K_0 of this point there exists an analytic function $\varphi(z)$ coinciding with $1/(1-z)$ in the part common to K_0 and G. But this means there is a finite limit

$$\lim_{z\to 1,\, z\in K\cap G} \varphi(z) = \lim_{z\to 1} \frac{1}{1-z} = \varphi(1),$$

which, obviously, cannot be true. Hence, point 1 is the singular point of $1/(1-z)$ (namely, the pole).

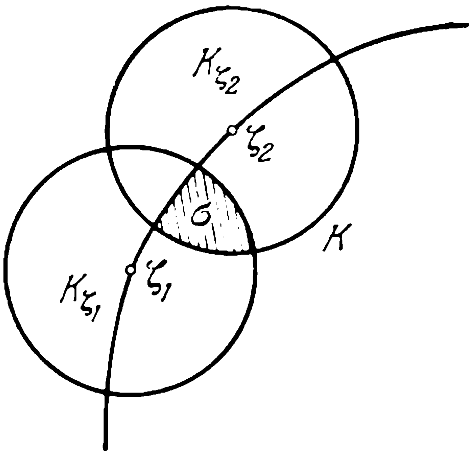


Fig. 66

Theorem. *Let* $\sum_0^\infty a_n (z-z_0)^n = f(z)$ *be a power series with a finite radius of convergence R. Then the sum of this series has at least one singular point on the boundary Γ of the convergence circle K.*

If we assume that the above statement is false, each point ζ on Γ must be a regular point of $f(z)$. For each such point we can then build a neighborhood K_ζ and in it an analytic function $\varphi_\zeta(z)$ coinciding with $f(z)$ in the part common to K_ζ and K. The union of K and all circles K_ζ represents a domain D to which $f(z)$ is analytically continued. Let us show that the function obtained as a result of such continuation is single-valued. Indeed, suppose that K_{ζ_1} and K_{ζ_2} have a common part, a lune (Fig. 66). Then there must exist a part σ of the lune common to three circles, K, K_{ζ_1} and K_{ζ_2} (the hatched section in Fig. 66). Since the values of each function $\varphi_{\zeta_1}(z)$ and $\varphi_{\zeta_2}(z)$ coincide in σ with those of $f(z)$, we see that $\varphi_{\zeta_1}(z) = \varphi_{\zeta_2}(z)$, $z \in \sigma$, and by the uniqueness theorem the same is true

for every point of the lune. Thus, if a point of D belongs to two different circles K_{ζ_1} and K_{ζ_2}, taking the values of the continued functions $\varphi_{\zeta_1}(z)$ or $\varphi_{\zeta_2}(z)$ at this point we arrive at the same number. This reasoning, applied to each pair of circles from $\{K_\zeta\}$, implies that the continued function $f(z)$ is single-valued. We also note that each point $\zeta \in \Gamma$ is an interior point for D, whence the distance R' from point z_0 to the boundary Δ of D is greater than R. But then the series that represents $f(z)$ in powers of $z - z_0$ (it cannot differ from the given series) must converge in the circle $|z - z_0| < R'$, which contradicts the hypothesis of the theorem. The proof is complete.

Corollary. *For the convergence radius R of the Taylor expansion for a function $f(z)$,*

$$f(z) = f(z_0) + \frac{f'(z_0)}{1!}(z - z_0) + \ldots + \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n + \ldots, \quad (9.3)$$

a function that is analytic in a circle K: $|z - z_0| < \rho$, to be equal to the radius ρ of this circle, it is necessary and sufficient that there be at least one singular point of the element $\{K, f(z)\}$ on the boundary of the circle, Γ: $|z - z_0| = \rho$.

Indeed, if $R = \rho$, then by the above theorem there is at least one singular point on Γ. Whence the necessity for R and ρ to be equal. But it is also sufficient. Indeed, by the theorem of Sec. 6.2, $R \geqslant \rho$. If we assume that there is at least one singular point on Γ and that $R \neq \rho$, we must have $R > \rho$. In this case the sum of the power series (9.3) is represented by a function analytic in the circle $|z - z_0| < R$ and, hence, analytic in a neighborhood of each point on Γ and coinciding with $f(z)$ in the interior of Γ. This implies that each point of Γ is regular for $f(z)$, a contradiction that proves the above proposition.

As an example we take the geometric series

$$1 + z + z^2 + \quad + z^n +$$

Its convergence radius is unity and its sum is $1/(1 - z)$. On the boundary of the convergence circle, $|z| = 1$, there is only one singular point $z = 1$, as we have already seen.

There is no difficulty in finding other power series for which each point on the boundary of the convergence circle is a singular point of the sum. Here is one simple example of such a series.

Consider the series

$$f(z) = 1 + z^2 + z^4 + \cdots + z^{2^n} + \cdots$$

Its convergence radius is, obviously, unity. We wish to show that as z tends to unity from inside the unit circle along its radius (i.e. along the real axis), $f(z)$ tends to ∞. Indeed, for each positive integer n the partial sum of the series, $1 + x^2 + \ldots + x^{2^n}$, tends to

$n+1$ as $x \to 1$ and, therefore, satisfies the inequality

$1+x^2+\ldots+x^{2^n} > n$ for $1-x<\delta(n)$, i.e. $x > 1-\delta(n)$.

But for the same values of x we have

$$f(x)=\sum_0^\infty x^{2^k} > \sum_0^n x^{2^k} > n,$$

which implies that

$$\lim_{x\to 1} f(x)=\infty.$$

On the basis of this fact we can easily see that just as in the case of a geometric series point 1 is a singular point of $f(z)$.

We write the identity

$$f(z)=z^2+z^4+\ldots+z^{2^n}+[1+(z^{2^n})^2+(z^{2^n})^4+\ldots].$$

Since the series in the brackets differs from the initial one only in that z^{2^n} is substituted for z, we conclude that

$$f(z)=z^2+z^4+\ldots+z^{2^n}+f(z^{2^n})$$

for any positive integer n.

Let us consider the 2^nth roots of unity: $\sqrt[2^n]{1}$. They are represented by points on the unit circle at the vertices of a regular 2^n-gon.

If ζ is one of these points and z is a point of the unit circle that lies on the radius $O\zeta$, point z^{2^n} obviously lies on the radius $O1$, and $z^{2^n}\to 1$ as $z\to\zeta$. This implies that

$$\lim_{\substack{z\to\zeta\\ z\in O\zeta}} f(z^{2^n})=\infty$$

and, hence,

$$\lim_{\substack{z\to\zeta\\ z\in O\zeta}} f(z)=\lim_{\substack{z\to\zeta\\ z\in O\zeta}}[z^2+z^4+\ldots+z^{2^n}+f(z^{2^n})]=\infty.$$

Thus, each of the roots of $\sqrt[2^n]{1}$ is also a singular point of $f(z)$ (for $n=1, 2, 3, \ldots$). We see that the singular points of $f(z)$ form a set everywhere dense on the unit circle (i.e. in a way such that any arbitrarily small arc of the circle contains elements of this set).

But from this it follows that all points on the unit circle without exception are singular points of $f(z)$, because if there is a regular point on the unit circle, there is also an arc all of whose points are regular, which in the given case is impossible.

9.7

CRITERION FOR DETECTING SINGULAR POINTS

Here is a general method that enables us to decide for any point ζ lying on the boundary Γ of the convergence circle of the power series (9.3) whether this point is regular or singular for the sum $f(z)$ of the series. Suppose that z_1 is a point on the radius $z_0\zeta$ differing from z_0 and ζ. We expand $f(z)$ in powers of $z - z_1$. We obtain

$$f(z) = b_0 + b_1(z - z_1) + \dots + b_n(z - z_1)^n + \dots, \qquad (9.4)$$

where

$$b_n = \frac{f^{(n)}(z_1)}{n!} = a_n + \frac{n+1}{1} a_{n+1}(z_1 - z_0) + \frac{(n+1)(n+2)}{1\times 2} a_{n+2}(z_1 - z_0)^2 + \dots \quad (n = 0, 1, 2, \dots).$$

By the theorem of Sec. 6.2, the series converges in the circle $|z - z_1| < \Delta$, where Δ is the distance from z_1 to Γ, i.e. $\Delta = R - |z_1 - z_0|$.

Hence, the series (9.4) converges inside the circumference γ centered at point z_1, a circumference that touches Γ at point ζ. According to the Cauchy-Hadamard formula, the radius of convergence of (9.4) is

$$r = \frac{1}{\overline{\lim\limits_{n\to\infty}} \sqrt[n]{|b_n|}}.$$

If r coincides with Δ, then on γ: $|z - z_0| = \Delta$ there must be at least one singular point of the sum of the series (9.4). But there cannot be a single point $\zeta' \in \gamma$ lying in the interior of K that is singular for the sum because in a neighborhood of ζ' that entirely belongs to K the function $f(z)$ is analytic, and inside γ it coincides with the sum of (9.4). Consequently, the singular point is ζ. Obviously, it must also be singular for $f(z)$, the sum of the series (9.3). If we were to assume the contrary, we would have a function $\varphi(z)$ analytic in a neighborhood of ζ such that at points of the neighborhood lying in K it would coincide with $f(z)$. Then the same function would coincide with the sum of (9.4) at points of the neighborhood inside γ, i.e. point ζ would not be singular for (9.4).

Thus, if

$$\Delta = R - |z_1 - z_0| = r = \frac{1}{\overline{\lim\limits_{n\to\infty}} \sqrt[n]{|b_n|}},$$

point ζ is singular for $f(z)$. We wish to show that when $\Delta \neq r$, i.e. $\Delta < r$, point ζ is regular for $f(z)$.

Indeed, in this case the sum of the series (9.4) represents a function $\varphi(z)$ analytic in a neighborhood U_ζ of point ζ and coinciding with $f(z)$ in the part of U_ζ that lies inside γ (Fig. 67). But $f(z)$ and $\varphi(z)$ are single-valued and analytic in the lune, which is the part common to U_ζ and K. From the fact that they coincide in the hatched section of the lune it follows, by the uniqueness theorem, that $\varphi(z)$ coincides with $f(z)$ in the entire lune. Hence, point ζ in this case is regular for $f(z)$.

We have therefore established that *point ζ is singular or regular for $f(z)$ depending on whether*

$$\Delta = R - |z_1 - z_0| = \frac{1}{\overline{\lim\limits_{n\to\infty}} \sqrt[n]{|b_n|}} = \frac{1}{\overline{\lim\limits_{n\to\infty}} \sqrt[n]{\dfrac{|f^{(n)}(z_1)|}{n!}}}$$

or

$$\Delta < \frac{1}{\overline{\lim\limits_{n\to\infty}} \sqrt[n]{\dfrac{|f^{(n)}(z_1)|}{n!}}}.$$

As an illustration of the above criterion we prove

Pringsheim's theorem (1894). *If the coefficients in $\sum\limits_0^\infty a_n z^n$ with a unit convergence circle are nonnegative real numbers $a_n \geqslant 0$, point $z = 1$ is singular for the sum of this series.*

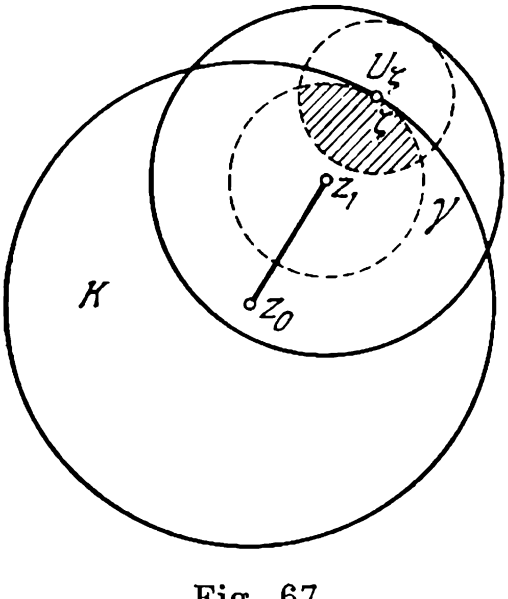


Fig. 67

To prove this proposition, we take a point $z_1 = x$ on the $O1$ radius. If we assume, for a moment, that point 1 is not singular for the sum of the series, then according to what we have just proved we must have

$$\Delta = R - |z_1 - z_0| = 1 - x < \frac{1}{\overline{\lim\limits_{n\to\infty}} \sqrt[n]{\dfrac{|f^{(n)}(x)|}{n!}}}. \tag{9.5}$$

Now let us take an arbitrary point ζ on the unit circle; suppose that z_1 is a point on radius $O\zeta$ that lies on the circle $|z| = x$, i.e. $|z_1| = x$. Then the distance Δ from z_1 to the unit circle is $1 - x$. On the other hand,

$$|f^{(n)}(z_1)| = \left| a_n + \frac{n+1}{1} a_{n+1} z_1 + \frac{(n+1)(n+2)}{2!} a_{n+2} z_1^2 + \ldots \right|$$
$$\leqslant a_n + \frac{n+1}{1} a_{n+1} x + \frac{(n+1)(n+2)}{2!} a_{n+2} x^2 + \ldots = f^{(n)}(x)$$

and, consequently,

$$\frac{1}{\overline{\lim\limits_{n\to\infty}} \sqrt[n]{\frac{|f^{(n)}(z_1)|}{n!}}} \geqslant \frac{1}{\overline{\lim\limits_{n\to\infty}} \sqrt[n]{\frac{f^{(n)}(x)}{n!}}}. \tag{9.6}$$

Whence on the basis of (9.5) and (9.6) we have at point z_1 the following inequality:

$$\Delta < \frac{1}{\overline{\lim\limits_{n\to\infty}} \sqrt[n]{\frac{|f^{(n)}(z_1)|}{n!}}},$$

which implies that for the sum of the series $\sum\limits_0^\infty a_n z^n$ all points on the unit circle are regular, a fact that contradicts the hypothesis of Pringsheim's theorem (that the unit circle is the boundary of the circle of convergence).

Thus, the point $z = 1$ must be singular for the sum of the above series provided that $a_n \geqslant 0$ and $R = 1$.

The proof shows that if instead of point 1 we take any other point ζ on the unit circle, for ζ to be a singular point we only have to require that the numbers $a_n \zeta^n$ be real and nonnegative. Moreover, it suffices to require that these numbers be real and nonnegative only starting from a number $n \geqslant n_0$, since if we write $f(z)$ in the form

$$f(z) = \sum_0^{n_0-1} a_k z^k + \sum_{n_0}^\infty a_k z^k,$$

we immediately see that the point ζ is singular for $f(z)$ if and only if it is a singular point of the sum of the series $\sum\limits_{n_0}^\infty a_k z^k$.

Consider, for instance, the series $\sum\limits_0^\infty \frac{z^{2^k}}{2^{k^2}}$. Here the coefficients a_n are zero if $n \neq 2^k$ and $1/2^{k^2}$ if $n = 2^k$. Consequently

$$\overline{\lim_{n\to\infty}} \sqrt[n]{|a_n|} = \lim_{k\to\infty} \sqrt[2^k]{\frac{1}{2^{k^2}}} = 1,$$

whence from the Cauchy-Hadamard formula it follows that the convergence radius is unity.

Then, by the proved theorem, the point $z = 1$ is singular for the sum $f(z)$ of the series. But the same theorem implies (due to the remark made above) that each point $\zeta = \sqrt[2^n]{1}$, where n is an arbitrary positive integer, is a singular point of $f(z)$. Indeed, at $k \geqslant n$ we have

$$\frac{\zeta^{2^k}}{2^{k^2}} = \frac{(\zeta^{2^n})^{2^{k-n}}}{2^{k^2}} = \frac{1}{2^{k^2}} > 0.$$

Hence, the set of singular points of $f(z)$ is everywhere dense on the unit circle. This means that there are no regular points of $f(z)$ on this circle, i.e. all points ζ ($|\zeta| = 1$) are singularities.

It is significant that this does not interfere with the fact that the given series converges absolutely and uniformly in the closed unit circle and that its sum $f(z)$ is an infinitely differentiable function on the set of values of z such that $|z| \leqslant 1$ (in particular, at all singular points). Indeed, for $|z| \leqslant 1$ we have

$$\left|\frac{z^{2^k}}{2^{k^2}}\right| \leqslant \frac{1}{2^{k^2}},$$

and since the series $\sum\limits_0^\infty \frac{1}{2^{k^2}}$ converges, the series $\sum\limits_0^\infty \frac{z^{2^k}}{2^{k^2}}$ converges absolutely and uniformly in the closed circle and, hence, its sum $f(z)$ is continuous in the same circle. Next, if we differentiate the given series m times termwise, we arrive at a series

$$\sum_0^\infty \frac{2^k(2^k-1)\ldots(2^k-m+1)}{2^{k^2}} z^{2^k-m}$$

with the moduli of the terms at $|z| \leqslant 1$ and $k > m$ satisfying the following inequalities:

$$\left|\frac{2^k(2^k-1)\ldots(2^k-m+1)}{2^{k^2}} z^{2^k-m}\right| \leqslant \frac{2^{km}}{2^{k^2}} = \frac{1}{2^{k(k-m)}} < \frac{1}{2^k}.$$

Consequently, all series obtained by termwise differentiation of the power series $\sum\limits_0^\infty \frac{z^{2^k}}{2^{k^2}}$ converge uniformly in the closed circle $|z| \leqslant 1$. whence a theorem known from the theory of functions of a real variable and extended to the theory of functions of a complex variable implies that these series represent the derivatives $f^{(m)}(z)$. Hence, $f(z)$ is continuous and infinitely differentiable in the closed circle $|z| \leqslant 1$ and analytic in the interior of the circle, and each point on the circle is a singular point.

This instructive example shows that the presence of singular points of an analytic function on the boundary of a given domain (circle) may have no effect in some cases on the behavior of this function near a singular point. More precisely, the function may not exhibit any breakdown in its continuity or the continuity of its derivatives at a given boundary point ζ. The comparison of $\left[\overline{\lim} \sqrt[n]{\frac{f^{(n)}(z_1)}{n!}}\right]^{-1}$

with the distance Δ from a point z_1 in the given domain to ζ determines whether ζ is a singular or regular point of $f(z)$ only if we consider all the derivatives of $f(z)$ at z_1.

9.8

DETERMINING THE CONVERGENCE RADIUS OF A POWER SERIES FROM THE DISTRIBUTION OF SINGULARITIES

The theorem and corollary of Sec. 9.6 often make it possible to find the convergence radius of the Taylor expansion for an analytic function $f(z)$ without calculating the expansion coefficients, i.e. the values of $f^{(n)}(z)/n!$. What needs to be done is to find the singular points of the elements. We must bear in mind here that to a given function $f(z)$ analytic in a domain G there corresponds an infinitude of elements because of the infinitude of circles with different centers belonging to the domain. In this way we are forced to deal with singular points of different elements of the same analytic function, and it is possible that a point singular for one element may be regular for another. Here are some simple examples illustrating this fact, but first let us note that when the analytic function is given in terms of a finite number of elementary functions, the possible singular points of its elements can easily be found among its points of discontinuity, for instance, points at which the function becomes infinite, and also among the branch points of $f(z)$.

Examples. (a) Take $f(z) = 1/(1 + z^2)$. This function is single-valued and analytic in the entire plane except at points $z = \pm i$, where it is infinite. Let z_0 be an arbitrary point distinct from $\pm i$, we take it as the center of the circle γ: $|z - z_0| = \rho$ that passes through one of the two points $\pm i$ closest to z_0. For the sake of definiteness we assume it to be i. The function $f(z)$ is analytic in the interior of γ and, hence, represents an element. Let us see whether i is a singular point of this element. If we assume that the contrary is true, we must have a neighborhood U of i and in it an analytic function $\varphi(z)$ that coincides with $f(z)$ in the part of U lying inside γ (we denote this part by d). Then at i there exists the finite limit

$$\varphi(i) = \lim_{\substack{z \to i \\ z \in d}} \varphi(z) = \lim_{\substack{z \to i \\ z \in d}} f(z) = \lim_{\substack{z \to i \\ z \in d}} \frac{1}{1+z^2},$$

which is obviously impossible.

Hence, γ contains a singular point of $f(z)$ and, hence, the convergence radius R of the Taylor expansion of $f(z)$ in powers of $z - z_0$ coincides with the radius ρ of γ, which is simply the distance between one of the two points $\pm i$ closest to z_0 and this point.

The simplest way to find the Taylor expansion in this case is to expand $1/(1+z^2)$ in partial fractions and employ the geometric series. This yields

$$\frac{1}{1+z^2}=-\frac{1}{2i}\left(\frac{1}{i-z}+\frac{1}{i+z}\right)=-\frac{1}{2i}\left(\frac{1}{i-z_0}\,\frac{1}{1-\frac{z-z_0}{i-z_0}}+\frac{1}{i+z_0}\,\frac{1}{1+\frac{z-z_0}{i+z_0}}\right).$$

Since $\left|\frac{z-z_0}{i-z_0}\right|<1$ and $\left|\frac{z-z_0}{i+z_0}\right|<1$ (point z lies in the interior of γ and points $\pm i$ either both on γ or one on γ and the other in the exterior of γ), each of the fractions $\frac{1}{1-\frac{z-z_0}{i-z_0}}$ and $\frac{1}{1+\frac{z-z_0}{i+z_0}}$ can be represented by a geometric series with a ratio $\frac{z-z_0}{i-z_0}$ or $-\frac{z-z_0}{i+z_0}$. We have

$$\frac{1}{1+z^2}=-\frac{1}{2i}\left[\frac{1}{i-z_0}\sum_0^\infty\left(\frac{z-z_0}{i-z_0}\right)^n+\frac{1}{i+z_0}\sum_0^\infty(-1)^n\left(\frac{z-z_0}{i+z_0}\right)^n\right]$$

$$=\frac{1}{2i}\sum_0^\infty(-1)^{n+1}\left[(z_0+i)^{-n-1}-(z_0-i)^{-n-1}\right](z-z_0)^n.$$

This is the expansion whose convergence radius R we determined earlier. In particular, for $z_0=1$ we have $R=\rho=\sqrt{2}$, and the expansion is

$$\frac{1}{1+z^2}=\sum_0^\infty(-1)^n\,2^{-(n+1)/2}\left[\sin(n+1)\frac{\pi}{4}\right](z-1)^n$$

$$=\frac{1}{2}-\frac{1}{2}(z-1)+\frac{1}{4}(z-1)^2-\frac{1}{8}(z-1)^4+\dots.$$

(b) Take $F(z)=1/\mathrm{Ln}\,z$. We consider a domain G whose boundary is the nonnegative part of the real axis, Λ: $x\geqslant 0$, $y=0$. In this domain the multiply valued function $F(z)$ separates into branches, of which we will consider the one with

$$f(z)=\frac{1}{\mathrm{Ln}_1 z}=\frac{1}{\ln|z|+i\,\mathrm{Arg}_1 z},$$

where

$$0<\mathrm{Arg}_1 z<2\pi.$$

From the definition of $F(z)$ (or $f(z)$) it follows that the singular points of the elements of $F(z)$ may coincide with either $z=0$ (the

branch point for Ln z and $F(z)$) or $z = 1$ (a point at which one of the values of the logarithm is zero). For the sake of definiteness we put $z_0 = 1 + i$. Then of the two points 0 and 1 the point 1 is the closest to z_0. Drawing a circle γ with its center at z_0 and a radius ρ equal to unity (Fig. 68), we find that in the interior of γ the function $f(z)$ determines an element $\varphi(z)$. Since both $\ln |z|$ and $\mathrm{Arg}_1 z$ tend to zero as z (in the interior of γ) tends to point 1 on γ, the values of $\varphi(z)$ tend to infinity. Whence, arguing in the same manner as we did in the previous example, we conclude that $z = 1$ is a singular point of the element. Therefore, the convergence radius R of the Taylor expansion for $\varphi(z)$ (in powers of $z-(1+i)$) is unity.

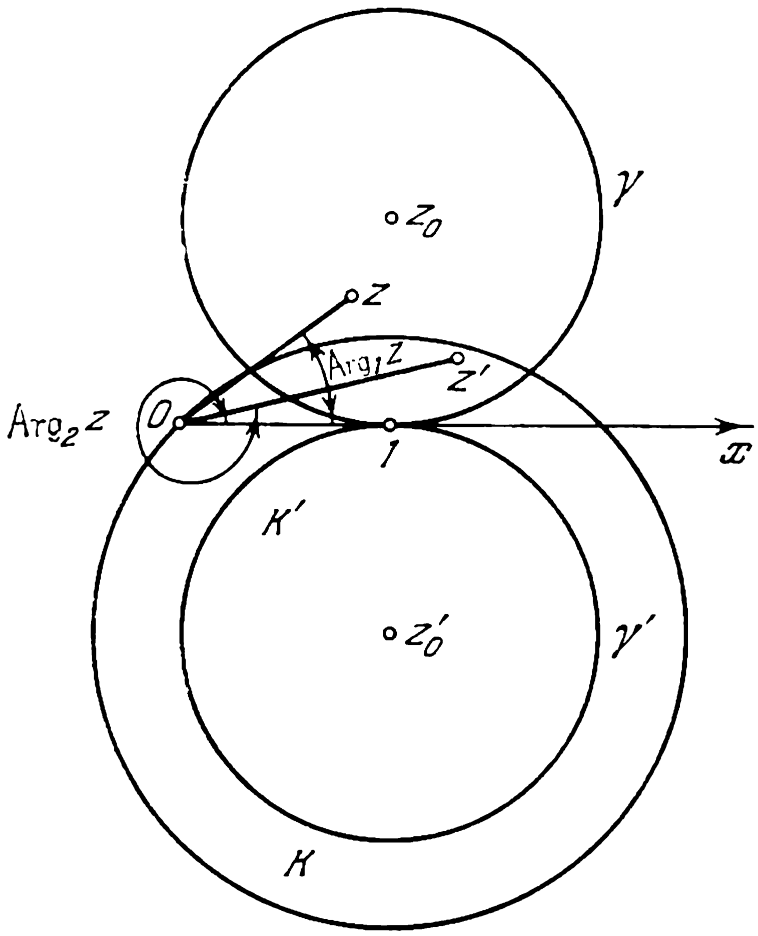


Fig. 68

Let us now consider the point $z_0' = 1 - i$, which lies in the lower half-plane. Of the two points 0 and 1 point 1 is once more closest to z_0. Drawing a circle γ' with its center at z_0' and a radius equal to unity, we find that in the interior of γ' there is an element of $f(z)$, which we denote by $\psi(z)$. We wish to show that all points of γ' are regular for this element. To this end we consider a circle K: $|z-(1-i)| < \sqrt{2}$, whose boundary contains point 0. In K the function Ln z separates into branches, one of which coincides with $\mathrm{Ln}_1 z$ inside γ' (it suffices to take the branch of Ln z whose value at z_0' coincides with $\mathrm{Ln}_1 z$). We denote this branch of Ln z by $\mathrm{Ln}_2 z = \ln|z| + i\,\mathrm{Arg}_2 z$. Since at z_0' the value of $\mathrm{Arg}_2 z$ coincides with that of $\mathrm{Arg}_1 z$ equal to $7\pi/4$ and in K the values of $\mathrm{Arg}_2 z$ differ in absolute value from that at the center by less than $\pi/2$, for all points in K we have

$$\frac{5\pi}{4} < \mathrm{Arg}_2 z < \frac{9\pi}{4}.$$

This implies that $\mathrm{Ln}_2 z$ does not vanish in K, whence $1/\mathrm{Ln}_2 z$ is analytic in K. But in the circle K': $|z-(1-i)| < 1$ this function coincides with $1/\mathrm{Ln}_1 z$, i.e. with the element $\psi(z)$ of $f(z)$. It follows then that each point on γ' is regular for $\psi(z)$ (indeed, for

each point $\zeta \in \gamma'$ there is a neighborhood U_ζ and in it the analytic function $1/\mathrm{Ln}_2 z$, which coincides with $\psi(z)$ in the part common to U_ζ and K'). In particular, point 1 is also regular for $\psi(z)$. This example shows that the same point 1 can be singular for one element of an analytic function $f(z)$ (for $\varphi(z)$) and regular for another (for $\psi(z)$).

The fact that all points on γ' are regular for $\psi(z)$ implies that the convergence radius of the Taylor expansion in powers of $z-(1-i)$ for this element is greater than the radius of γ', which is unity. But $\psi(z) = \mathrm{Ln}_2 z$ in K', whence the Taylor expansions for $\psi(z)$ and $\mathrm{Ln}_2 z$ coincide. Since $\mathrm{Ln}_2 z$ is a function analytic in the circle $|z-(1-i)| < \sqrt{2}$, the convergence radius R' of these expansions cannot be less than $\sqrt{2}$. To prove that it is exactly $\sqrt{2}$ it suffices to show that at least one point on the circle Γ: $|z-(1-i)| = \sqrt{2}$ is singular for $1/\mathrm{Ln}_2 z$. The origin of coordinates is just such a point. Indeed, $(1/\mathrm{Ln}_2 z)' = -1/z(\mathrm{Ln}_2 z)^2 \to \infty$ as $z \to 0$ remaining in the interior of Γ. This means that there is not a single function $\chi(z)$ analytic in a neighborhood U of the origin of coordinates that would coincide with $1/\mathrm{Ln}_2 z$ at points common to U and K. (If we assume that such a function exists, then we must also assume that there is a finite limit

$$\chi'(0) = \lim_{z\to 0} \chi'(z) = \lim_{z\to 0}\left(\frac{1}{\mathrm{Ln}_2 z}\right)',$$

which, of course, is impossible.)

The reader is advised to verify that in this example for each point z_0 of G lying in the upper half-plane the convergence radius of the Taylor expansion in powers of $z-z_0$ for $f(z)$ is equal to the distance from z_0 to the one of the two points 0 and 1 closest to it, while for points z_0' in the lower half-plane the distance from z_0' to 1 plays no role in determining the convergence radius of the corresponding series; this radius always coincides with the distance from z_0' to point 0.

9.9

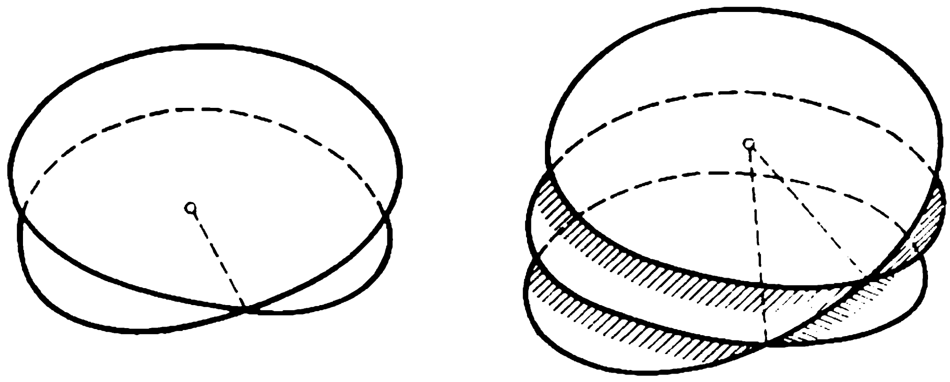
BRANCH POINTS

Let us apply the results of the previous sections to a study of the behavior of functions in a neighborhood of an isolated singular point of a special nature, the branch point. Let z_0 be a (finite) point of the plane and G_j $(j = 0, \pm 1, \pm 2, \ldots)$ alternately the upper and lower semicircles of a radius $R \leqslant \infty$ centered at this point. Precisely, by introducing the polar coordinates φ and r with the origin at z_0 we will determine the G_j by means of the inequalities $(j-1)\pi < \varphi < j\pi$, $0 < r < R$ (obviously, the upper semicircle is realized for odd j's and the lower for even j's). Suppose that

$\{G_j, f_j(z)\}$ are elements such that for $j = 0, \pm 1, \pm 2, \ldots$ the function $f_j(z)$ is continuable across the radius $\varphi = j\pi$, $0 < r < R$, to G_{j+1} and $f_{j+1}(z)$ is the result of such continuation. Then the set of elements $\{G_j, f_j(z)\}$ determines in the neighborhood $0 < < |z - z_0| < R$ of point z_0 an analytic function $f(z)$. If $\{G_3, f_3(z)\}$ coincides with $\{G_1, f_1(z)\}$, we can easily see that each element $\{G_{j+2}, f_{j+2}\}$ coincides with $\{G_j, f_j\}$ $(j = 0, \pm 1, \pm 2, \ldots)$ and the function $f(z)$ is single-valued. In the upper semicircle G_1 it coincides with $f_1(z)$ and in the lower G_2 with $f_2(z)$; continuation of $f_1(z)$ across the radius $\varphi = \pi$, $0 < r < R$, to the lower semicircle yields $f_2(z)$, and continuation of $f_2(z)$ across the radius $\varphi = 2\pi$, $0 < r < R$, to the upper semicircle yields $f_1(z)$. Point $z = z_0$ may be either regular for $f(z)$ or an ordinary isolated singularity of $f(z)$. This case was studied extensively in Chap. 7. Now let us assume that $\{G_3, f_3(z)\}$ does not coincide with $\{G_1, f_1(z)\}$, i.e. $f_3(z)$ differs from $f_1(z)$ in the upper semicircle. Then we turn to the elements $\{G_5, f_5(z)\}, \ldots, \{G_{2k+1}, f_{2k+1}(z)\}$ and compare each of them with $\{G_1, f_1(z)\}$. There are only two possibilities here: either among the infinitude of elements $\{G_{2k+1}, f_{2k+1}\}$ $(k > 1)$ there is an element that coincides with $\{G_1, f_1(z)\}$ or there is no such element. We consider the first possibility. Suppose that $\{G_{2k_0+1}, f_{2k_0+1}\}$ $(k_0 > 1)$ is the element with the smallest subscript that coincides with $\{G_1, f_1(z)\}$. We wish to show that then $\{G_{j+2k_0}, f_{j+2k_0}\}$ and $\{G_j, f_j\}$ coincide for any positive integer j. It obviously suffices to show that the fact that $\{G_{j_0+2k_0}, f_{j_0+2k_0}\}$ and $\{G_{j_0}, f_{j_0}\}$ coincide for a positive integer j_0 implies that $\{G_{j_0\pm 1+2k_0}, f_{j_0\pm 1+2k_0}\}$ and $\{G_{j_0\pm 1}, f_{j_0\pm 1}\}$ coincide, too. But if the analytic functions $f_{j_0}(z)$ and $f_{j_0+2k_0}(z)$ coincide in the semicircle $G_{j_0} = G_{j_0+2k_0}$, then the results of their analytic continuation across the same radius to the other semicircle $G_{j_0+1} = G_{j_0+1+2k_0}$ coincide, too (in view of the uniqueness theorem). Similarly, the assumption that the functions $f_{j_0-1}(z)$ and $f_{j_0-1+2k_0}(z)$ do not coincide in the semicircle $G_{j_0-1} = G_{j_0-1+2k_0}$ would lead to a situation in which $f_{j_0}(z)$ and $f_{j_0+2k_0}(z)$, which are the results of analytic continuation of the former functions across the same radius to the semicircle $G_{j_0} = G_{j_0+2k_0}$, would not coincide. This reasoning shows that if $\{G_1, f_1(z)\}$ and $\{G_{1+2k_0}, f_{1+2k_0}(z)\}$ coincide, so do $\{G_j, f_j(z)\}$ and $\{G_{j+2k_0}, f_{j+2k_0}(z)\}$ for any j. Hence, among the infinitude of elements $\{G_n, f_n(z)\}$ there can only be a finite number of different elements: $\{G_1, f_1(z)\}, \ldots, \{G_{2k_0}, f_{2k_0}(z)\}$. We can easily see that they are all different. Suppose that the contrary is true, and let the elements $\{G_p, f_p(z)\}$ and $\{G_q, f_q(z)\}$, with $1 \leqslant p < q \leqslant 2k_0$, coincide. Employing the result established above, we find that for any integer j the elements $\{G_j, f_j(z)\}$ and $\{G_{j+q-p}, f_{j+q-p}(z)\}$ coincide, too. In particular, this must be the case for $\{G_1, f_1(z)\}$ and $\{G_{1+q-p}, f_{1+q-p}(z)\}$, with $0 < q - p < 2k_0$. But this contradicts the assump-

tion that $2k_0$ is the smallest of the positive integers n for which $\{G_{1+n}, f_{1+n}(z)\}$ coincides with $\{G_1, f_1(z)\}$.

Thus, the k_0 functions $f_1(z), f_3(z), \ldots, f_{2k_0-1}(z)$ are different in the semicircle G_1 just as the k_0 functions $f_2(z), \ldots, f_{2k_0}(z)$ are in G_2. This implies that the analytic function $f(z)$ defined by the elements $\{G_j, f_j(z)\}$ in the domain $0 < |z - z_0| < R$ is multiply valued, namely k_0-valued. Fixing a branch $f_j(z)$ of this function in one of the semicircles G_j and analytically continuing it to $G_{j+1}, G_{j+2}, \ldots$ across the respective radii, we obtain, after returning to the initial semicircle each time, new branches, $f_{j+2}(z)$, $f_{j+4}(z), \ldots$, until we reach the $f_{j+2k_0}(z)$ branch, which coincides with $f_j(z)$. This situation can be characterized as follows. If in



(a) (b)

Fig. 69

the course of analytic continuation we go around point z_0 counterclockwise, the element $\{G_j, f_j(z)\}$ transforms into a different element, $\{G_{j+2}, f_{j+2}(z)\}$; and only if we go around this point k_0 times will we arrive at the element $\{G_{j+2k_0}, f_{j+2k_0}(z)\}$, which coincides with the initial one. Point z_0 in this case is called a *branch point of* $f(z)$ *of finite multiplicity* (equal to k_0). It is easy to see that the Riemann surface for $f(z)$ in a neighborhood of point z_0 has the shape of a k_0-*sheeted circle* (Fig. 69a and b, where the cases depicted correspond to $k_0 = 2$ and $k_0 = 3$).

Let us find the analytic representation of $f(z)$ in the neighborhood of a branch point of multiplicity k_0 (degree of ramification $k_0 - 1$). To this end we subject the k_0-sheeted circle to the transformation $t = \sqrt[k_0]{z - z_0}$. Then the semicircles $G_1, G_2, \ldots, G_{2k_0}$ comprising the Riemann surface will transform into the sectors $g_1, g_2, \ldots, g_{2k_0}$ of a circle of radius $\underset{+}{\sqrt[k]{R}}$ with the center at the origin of coordinates; the g_j are defined by the inequalities $(j-1)\pi/k_0 < \alpha < j\pi/k_0$, $0 < \rho < \underset{+}{\sqrt[k]{R}}$ (α and ρ are polar coordinates, with the origin at $t = 0$). The function $f_j(z)$ is mapped into $f_j^*(t) =$

$= f_j(z_0 + t^{k_0})$, and $f_j^*(t)$ admits analytic continuation across the radius $\alpha = j\pi/k_0$, $0 < \rho < \sqrt[k_0]{R}$ (+), to the adjacent sector g_{j+1}; the result is $f_{j+1}^*(t)$. The elements $\{g_j, f_j^*(t)\}$ $(j = 1, 2, \ldots, 2k_0)$ define in the domain $0 < |t| < \sqrt[k_0]{R}$ (+) a single-valued analytic function $f^*(t)$, which can, therefore, be represented in the same domain by its Laurent expansion: $f^*(t) = \sum_{-\infty}^{+\infty} A_n t^n$. There are three possibilities for $A_{-1}, A_{-2}, \ldots, A_{-n}, \ldots$: (a) an infinite number of them may be nonzero, (b) only a finite number ($\geqslant 1$) may be nonzero, or (c) all may be zero. Case (a) takes place when $\lim_{t\to 0} f^*(t)$ does not exist, (b) when $\lim_{t\to 0} f^*(t) = \infty$, and (c) when $\lim_{t\to 0} f^*(t)$ exists and differs from ∞. Returning to z, we conclude that $f(z)$ can be expanded in integral powers of $(z - z_0)^{1/k_0}$:

$$f(z) = \sum_{-\infty}^{+\infty} A_n (z - z_0)^{\frac{n}{k_0}};$$

the coefficients with negative subscripts satisfy condition (a), (b), or (c) depending on whether $\lim_{z\to z_0} f(z)$ does not exist, is equal to infinity, or is finite.

In the last two cases (i.e. when $\lim_{z\to z_0} f(z)$ is finite or infinite) z_0 is called an *algebraic branch point*.

But if $\lim_{z\to z_0} f(z)$ does not exist, the branch point of multiplicity k_0 is called a *transcendental branch point of finite multiplicity* (equal to k_0). We can easily see that the functions $\sqrt[k_0]{z - z_0}$, $1/\sqrt[k_0]{z - z_0}$, $e^{\sqrt[k_0]{z - z_0}}$, and $\sin \sqrt[k_0]{z - z_0}$ have at $z = z_0$ an algebraic branch point of multiplicity k_0, and the functions $e^{1/\sqrt[k_0]{z - z_0}}$ and $\sin\left(1/\sqrt[k_0]{z - z_0}\right)$ have at $z = z_0$ a transcendental branch point of the same multiplicity k_0.

Returning to the general case, let us assume that there is not a single element $\{G_j, f_j(z)\}$ at $j > 1$ that coincides with $\{G_1, f_1(z)\}$. It then follows that there is no two elements $\{G_p, f_p(z)\}$ and $\{G_q, f_q(z)\}$, $p \neq q$, that are the same. In this case the analytic function $f(z)$ defined in the domain $0 < |z - z_0| < R$ by the collection of all elements $\{G_n, f_n(z)\}$ proves to be infinitely many-valued, and the analytic continuation of any element $\{G_j, f_j(z)\}$ in any number of revolutions about z_0 leads to an element that differs from the initial one. Point z_0 is then called a *logarithmic branch point*, or a *branch point of infinite multiplicity*. The Riemann surface for $f(z)$ in a neighborhood of z_0 has in this case the shape of an infi-

nite sheeted circle. Examples of functions with logarithmic branch points at z_0 are $\mathrm{Ln}\,(z-z_0)$ and $(z-z_0)^\alpha$ (α is not a rational real number). The function $\mathrm{Ln}\frac{z-a}{z-b}$ has two logarithmic points: $z=a$ and $z=b$; the function $\mathrm{Arc}\tan z=\frac{1}{2i}\,\mathrm{Ln}\,\frac{i-z}{i+z}$ also has two logarithmic branch points: $z=\pm i$.

The results of our discussion can also be applied to the point $z_0 = \infty$ if we take, say, the neighborhood $R<|z|<+\infty$ divided into two parts, the upper and lower half-planes. Then we only have to use the mapping $\zeta=1/z$ to reduce this case to that of the finite point $\zeta=0$.

We note that the name algebraic branch point is justified by the following

Theorem. *Among the various analytic functions only algebraic functions $F(z)$, i.e. functions that satisfy the equation $\sum\limits_{k=0}^{N} P_k(z)\times\times[F(z)]^k=0$ ($P_k(z)$ are polynomials, and $P_N(z)\not\equiv 0$, $N\geqslant 1$), have singular points in the extended plane that are exclusively poles and algebraic branch points.**

Examples. (a) The singular points of the algebraic function $\sqrt[3]{z^2-5z+6}$ are $z=2$, $z=3$, and $z=\infty$. Each is an algebraic branch point of multiplicity 3.

(b) The function $\sqrt[3]{1+z^3}$ has three algebraic branch points of multiplicity 3 at the points -1, $e^{\pi i/3}$, and $e^{-\pi i/3}$, and one simple pole at point ∞. Strictly speaking, in the neighborhood $|z|>1$ of point $z=\infty$ each of the three branches, namely $\sqrt[3]{1+z^3}=$

$$=z\sqrt[3]{1+z^{-3}}=ze^{2\pi ik/3}\left(1+\frac{1}{3}z^{-3}+\frac{\frac{1}{3}\left(\frac{1}{3}-1\right)}{1\times 2}z^{-6}+\dots\right),\quad\text{where}$$

$k=0$, 1, and 2, has a simple pole at $z=\infty$.

(c) The function $e^{\sqrt{z}}$ has an algebraic branch point of multiplicity 2 at point $z=0$ and a transcendental branch point of the same multiplicity at $z=\infty$.

(d) The function $\sqrt{\cos z}$ has algebraic branch points of multiplicity 2 at $z=\pi/2+k\pi$ ($k=0, \pm1, \pm2, \dots$) and a nonisolated branch point of the same multiplicity at $z=\infty$.

(e) $\sin\sqrt{z}/\sqrt{z}$, $\cosh\sqrt{z}$, and $e^{\sqrt[3]{z}}+e^{\varepsilon\sqrt[3]{z}}+e^{\varepsilon^2\sqrt[3]{z}}$, with $\varepsilon=e^{2\pi i/3}$ the cube root of unity, are entire functions (the single essential singularity being at infinity) because a revolution about a possible branch point (at the origin of coordinates or at the point at infinity) brings us to the initial values.

* See, for instance, A. I. Markushevich, *The Theory of Analytic Functions* [in Russian], vol. 2, Nauka, Moscow, 1968, Chap. 8, § 6.

9.10

ANALYTIC CONTINUATION ALONG A CURVE. THE MONODROMY THEOREM

Let L be a continuous curve with the initial point z_0 and the terminal point Z (the curve may be closed and then Z and z_0 coincide). Let us say that the element $\{g_0, f_0\}$, $z_0 \in g_0$ is *continuable along* L (or *to* L) if there is a chain of elements $\{g_k, f_k\}$, $k = 0, 1, 2, \ldots, n$, for which on L in the direction from z_0 to Z we can choose points $z_1, \ldots, z_n$ such that each arc $l_k \subset L$ with the initial point at z_k and the terminal point at z_{k+1} belongs to the domain g_k ($k = 0, 1, 2, \ldots, n$ and $z_{n+1} = Z$; see Fig. 70). When no such chain exists, $\{g_0, f_0\}$ is said to be *noncontinuable along* L.

Note that the same chain $\{g_k, f_k\}$ traversed in the opposite direction serves for continuing $\{g_n, f_n\}$ along curve $-L$, which differs from L only in its direction of traversal.

If the domains considered are circles, in the process of continuation along a curve it is usually required that the centers of the circles be on L in an order corresponding to the increasing subscripts of the elements, and that the center of the first circle be at z_0 while that of the last be at Z (though this requirement is not essential).

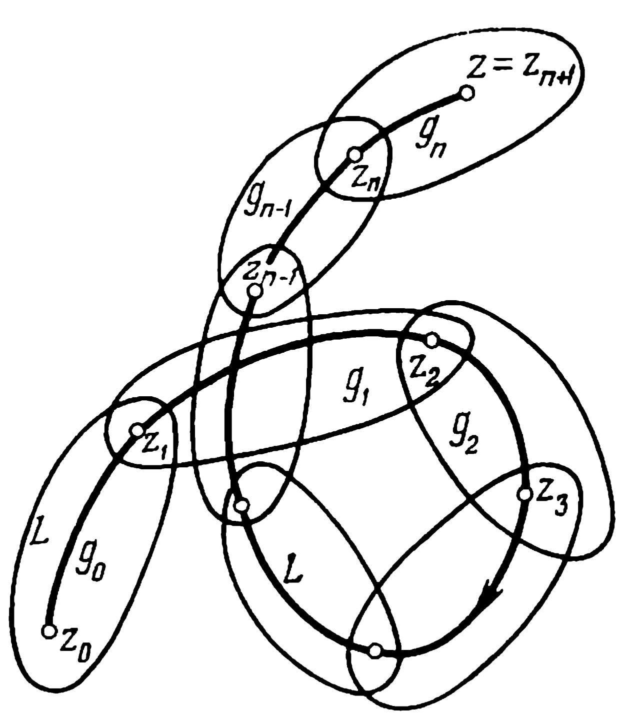


Fig. 70

In general, each continuation of $\{g_0, f_0\}$ to $\{g, f\}$ can be thought of as a continuation along a curve (a broken line, for one). Suppose that $\{g_k, f_k\}$, $k = 0, 1, 2, \ldots, n$, is the chain that connects $\{g_0, f_0\}$ with $\{g_n, f_n\} = \{g, f\}$. In g_0 and $g_n = g$ we choose a point, $z_0 \in g_0$ and $z_{n+1} \in g$, and, moreover, in each intersection $g_k \cap g_{k+1}$ a point z_{k+1} ($k = 0, 1, 2, \ldots, n - 1$). Connecting these points according to their subscripts by straight segments $z_k z_{k+1}$ ($k = 0, 1, 2, \ldots, n$), we arrive at a broken line Λ (the segment $z_k z_{k+1}$ belongs to g_k since g_k is convex). The studied continuation defines on Λ a function $F(z) = f_k(z)$, provided that $z \in z_k z_{k+1}$, $k = 0, 1, 2, \ldots, n$. The function may not be single-valued only at the points of self-intersection of Λ (if $z_l z_{l+1}$ meets $z_k z_{k+1}$ ($k < l$) at point η and $f_l(\eta) \neq f_k(\eta)$) or at point z_0 when the curve is closed ($z_0 = z_{n+1}$) and $f_0(z_0) \neq f_n(z_{n+1}) = f(z_{n+1})$.

Continuation along a curve enables us to somewhat generalize the idea of a singular point introduced in Secs. 9.6 and 9.9 for an

analytic function element. Let $\{g_0, f_0\}$ be noncontinuable along a continuous curve L with the initial point at $z_0 \in g_0$ and L be given by the equation $z = \lambda(t)$, $t_0 \leqslant t \leqslant T$. As long as t' is smaller than T and so close to t_0 that the arc $L_{t'}$, defined by the inequality $t_0 \leqslant t \leqslant t'$, belongs to g_0, the given element, obviously, remains continuable along $L_{t'}$ (here the chain consists of only one element, $\{g_0, f_0\}$). Therefore, there should be an upper bound for the numbers t' not greater than T and defined by the condition that $\{g_0, f_0\}$ is continuable along each arc $L_{t'} \subset L$ if $t' < \tau$ and is noncontinuable along arc L_τ. In this case the point $z = \lambda(\tau) \in L$ is a kind of an obstacle in the way of analytic continuation of $\{g_0, f_0\}$ along L. This point is called the *singular point* for the continuation considered.

In the light of the introduced ideas, the point on the boundary of the convergence circle of the power series

$$\sum_{0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n = f(z)$$

(the convergence radius R being finite) is regular or singular depending on whether the element $\{|z-z_0| < R, f(z)\}$ is or is not continuable along the radius with one of its end points at z_0. When the element is noncontinuable, this point is the obstacle in the way of analytic continuation along the given radius.

The reader must bear in mind that for a given element $\{g_0, f_0\}$ a point ζ in the plane may be regular in the continuation along one path and singular in the continuation along another.

Let us once more consider Example (b) of Sec. 9.8. In the circles K: $|z-(1+i)| < 1$ and K': $|z-(1-i)| < \sqrt{2}$ we can isolate the branches $\varphi(z)$ and $\psi(z)$, respectively, of $F(z) = 1/\mathrm{Ln}\, z = 1/(\ln z + i\theta)$ by requiring that $0 < \theta < \pi$ (for $\varphi(z)$) or $\pi < \theta < 2\pi$ (for $\psi(z)$). Since $\{K, \varphi(z)\}$ and $\{K', \psi(z)\}$ are elements of one analytic function $F(z)$, there exists a chain of elements connecting the two, and this chain therefore executes the continuation of the first element, for example, along a curve L. In the case at hand we can take an arc of the circle $|z| = \sqrt{2}$ defined via the inequality $\pi/4 \leqslant \theta \leqslant 7\pi/4$ as L (in Fig. 68 it would connect the center z_0 of K with the center z_0' of K' without intersecting the positive half of the real axis). Using the reasoning developed in Sec. 9.8 when solving this example, we discover that point $z = 1$ is a singular point in the continuation of $\{K, \varphi(z)\}$ along the radius of K directed at this point. But it is a regular point if we continue the same element along a curve L' obtained by joining a straight segment connecting points z_0' and 1 (the latter point lies in K') with L at its terminal point. The idea of continuation along a curve is simplified by the following

Monodromy theorem. *Let G be a simply connected domain in the finite plane and M the set of elements $\{g, f(z)\}$, $g \subset G$, of which any two are analytic continuations of each other along curves belonging to G. Suppose that M satisfies the following conditions:*

(1) *for each point $z_0 \in G$ and each continuous curve $L \subset G$ with the initial point at z_0, every element $\{g_0, f_0(z)\} \in M$ for which $z_0 \in g_0$ is continuable along L;*

(2) *if a closed curve L belongs to a sufficiently small neighborhood U_0 of point $z_0 \in G$, the continuation of each element $\{g_0, f_0(z)\} \in M$ along L defines a single-valued function on this curve.*

Then M defines a single-valued function in the entire domain G.

The term *monodromy* (from the Greek *monos* for single and *dromas* for running) was first used by Cauchy to denote single-valued functions of a complex variable. He elucidated that these functions behave in such a way that in a domain S they return to the same value irrespective of how the "point with affix z runs to and fro in S".

The monodromy theorem is of a purely topological nature. We will see from the proof that nowhere is the analyticity of a function used. It suffices to assume that for each element $\{g, f\} \in M$ the function $f(z)$ is simply single-valued and continuous in g. This generalization preserves the meaning of and significance of the notion of direct analytic continuation, a chain of direct continuations connecting any two elements, and the notion of continuation along a curve.

In the theory of analytic functions this theorem is usually used when M constitutes a set of circles obtained by continuing the same element $\{g_0, f_0\}$, where g_0 is a circle with its center at $z_0 \in G$, along every continuous curve emerging at this point and belonging to G. The hypothesis is then reduced to the element $\{g_0, f_0\}$ being continuable along each such curve.

Condition (1) is, obviously, met here. Indeed, let an element $\{g', f'\}$ centered at $z' \in G$ be noncontinuable along a curve L' that emerges at z'. Since we can arrive at this element by analytic continuation of $\{g_0, f_0\}$ along a curve L_0 that connects z_0 with z', the element $\{g_0, f_0\}$ is also noncontinuable along the composite curve L obtained by linking L_0 at its terminal point with L'. But this contradicts the assumed condition.

In view of the same assumption there is not a single boundary point of circle g, with $\{g, f\} \in M$, that is a singular point of $f(z)$, provided that it does not belong to the boundary of G; otherwise $\{g, f\}$ would be noncontinuable along a radius of g ending at this point. If we denote the center of g by z and the distance from z to the boundary of G by $\rho(z)$, we can modify the definition of M if for each element $\{g, f\}$ belonging to M we substitute its direct analytic continuation $\{g', f'\}$ to a circle with the same center and a radius $\rho(z)$.

Condition (2) in the monodromy theorem is now met if for the neighborhood U_0 of point $z_0 \in G$ we take the circle $|z - z_0| < \rho(z_0)/3$. Indeed, if $L \subset U_0$, then $\rho(z) > 2\rho(z_0)/3$ for each point $z \in L$, which is greater than the diameter of U_0. For this reason, if the center of circle g for an element $\{g, f\} \in M$ lies on L, then g contains U_0 and, hence, the entire curve L. But this means that the continuation along L is executed by only one element (to which all the chain is reduced), and the uniqueness of the continuation follows from that of $f(z)$ in g.

In view of these explanations the monodromy theorem can be applied to analytic functions in the following simplified form:

If G is a simply connected domain in the finite plane and $\{g, f\}$, $g \subset G$ is a circular element continuable along each continuous curve $L \subset G$ emerging at the center of g, then the function obtained as a result of all its possible continuations in G is unique.

Proof of the monodromy theorem. Suppose that the proposition is not true. Then in M there are two elements, $\{g_0, f_0\}$ and $\{g, f\}$, for which $\delta = g_0 \cap g$ is not empty and $f_0(\zeta) \neq f(\zeta)$ at some point $\zeta \in \delta$. If the $\{g_k, f_k(z)\}$ $(k = 0, 1, 2, \ldots, n)$ form a chain of elements (all from M) connecting $\{g_0, f_0\}$ with $\{g_n, f_n\} = \{g, f\}$, we can build the broken line Λ mentioned at the beginning of this section, along which this chain executes the continuation of $\{g_0, f_0\}$ to $\{g, f\}$; we take $z_0 = z_{n+1} = \zeta$.

Let us suppose that $F(z)$ is a function defined on Λ in the continuation. Under the assumption, it is not unique at the vertex $z_0 = z_{n+1}$ of Λ $(f_0(z_0) \neq f(z_{n+1}))$. We can arrive at the contradiction needed for the proof of the theorem by using the general line of reasoning in Cauchy's integral theorem (Secs. 5.4 and 5.5). Namely, for Λ we substitute polygonal contours that are more and more simple; these contours finally contract to a point in G, but they possess the same property: *at some vertex of each contour the function $F(z)$ obtained as a result of continuation along the contour is not unique.*

We start by freeing Λ from self-intersections (if such exist). Let us assume that in the traversal of Λ in the direction of increasing number of the vertex, starting from the vertex at z_0, a certain component $z_l z_{l+1}$ for the first time intersects a previously traversed component $z_k z_{k+1}$ $(k < l)$ at a point η. Then the part of Λ traversed in the same direction from η through z_{k+1} and z_l and ending at η constitutes a closed contour Λ' without self-intersections; the same is true for the other part of Λ traversed from z_0 to z_k to η and, bypassing Λ' to z_0 (cf. Sec. 5.4, Fig. 34, which illustrates a similar situation), and we denote this "complementary" contour by Λ''. If $f_k(\eta) \neq f_l(\eta)$, we substitute Λ' for Λ. But if $f_k(\eta) = f_l(\eta)$, we exclude Λ' from Λ and substitute Λ'' for Λ. We go on to the next point of self-intersection and eliminate it in the same manner. Obviously, after a finite number of steps we have a closed contour Λ_1

without self-intersections that possesses the same property as Λ printed in italics above.

Now we reduce Λ_1 to a triangle by reducing the number of vertices (provided that it is not a triangle from the start). Here we use Condition (1) of the theorem. To this end we draw a diagonal MN belonging to the interior of Λ_1 (with the exception of points M and N, see Sec. 5.4 and Fig. 33). The diagonal divides Λ_1 into two polygons Λ_1' and Λ_1'' with their number of vertices smaller than in Λ_1. If $M = z_k \in g_k$, with $\{g_k, f_k\}$ an element in the chain under consideration, Condition (1) states that we can build a new chain: $\{g_k, f_k\}$, $\{g_k', f_k'\}$, . . ., $\{g_k^{(\varkappa)}, f_k^{(\varkappa)}\}$ that establishes the continuation of $\{g_k, f_k\}$ along MN. There are two possibilities here: the function established as a result of the last continuation along MN has at point N either the same value as $F(z)$ did in the original continuation or a different value. In both cases at least one of the contours Λ_1' and Λ_1'' (we denote it by Λ_2) possesses the same property as Λ_1, namely, the function defined on it is not unique at one of its vertices. Substituting Λ_2 for Λ_1 and reasoning along the same lines, after a finite number of steps we arrive at a triangular contour Δ, a continuation along which is not unique at one of its vertices.

In conclusion we will infinitely refine the triangles by dividing them into four congruent triangles, as we did in Sec. 5.5 (see Fig. 35). The line of reasoning used above in relation to a diagonal of a polygon enables us here to isolate at least one of the four new triangles in which the nonuniqueness of the continuation at one of the vertices is preserved. We have arrived at a sequence of triangles $\{\Delta_n\}$ contracting to a point $\zeta_0 \in G$ (in our proof we constantly relied on the fact that G is simply connected).

But it is obvious that such a sequence contradicts Condition (2) of the theorem because the triangles Δ_n, starting from a definite one, will belong to the neighborhood U_0 of point ζ_0 mentioned in this condition. The proof of the monodromy theorem is complete.

9.11

ANALYTIC CONTINUATION AND THE BOREL TRANSFORMATION

In this section it is more convenient to write the power series as $\sum\limits_{n=0}^{\infty} \frac{c_n}{z^{n+1}}$. If $\overline{\lim} \sqrt[n]{|c_n|} = \sigma < +\infty$ (which from now on we assume to be true), this series converges at $|z| > \sigma$ and determines a single-valued analytic function $f(z)$. We will discuss the possibility of continuing this function analytically to the interior of the circle $|z| = \sigma$ (when σ is positive). Of course, such continuation

is possible only if not all points on the circle are singular for $f(z)$.

Suppose that $\zeta = \sigma e^{i\varphi}$ is a point on the circle. We draw a ray through this point that emerges at $z = 0$ and a straight line perpendicular to this ray; the points of the straight line are given by the equation $\operatorname{Re}(ze^{-i\varphi}) = \sigma$. The function $f(z)$ is analytic in the half-plane $\operatorname{Re}(ze^{-i\varphi}) > \sigma$ bounded by the above straight line and not containing the circle $|z| < \sigma$. In general, let us consider a half-plane of the type $\operatorname{Re}(ze^{-i\varphi}) > c$. The aforesaid implies that the lower bound of those c for which there is a direct analytic continuation of $f(z)$ in the appropriate half-plane is not greater than σ. On the other hand, this bound cannot be smaller than $-\sigma$. Indeed, if $c < -\sigma$, then the half-plane $\operatorname{Re}(ze^{-i\varphi}) > c$ contains the closed circle $|z| \leqslant \sigma$ and, hence, the singular points of $f(z)$ (at least one) lying on the circle $|z| = \sigma$. We denote the lower bound by $k(\varphi)$. Then $-\sigma \leqslant k(\varphi) \leqslant +\sigma$. In view of the definition of $k(\varphi)$ the sum of $\sum_0^\infty \frac{c_n}{z^{n+1}}$ admits direct analytic continuation to the half-plane $\operatorname{Re}(ze^{-i\varphi}) > k(\varphi)$, and we cannot substitute a smaller number for $k(\varphi)$ without violating this property.

Now let us assume that $z_0 = r_0e^{i\theta}$ is a point lying in the exterior of the circle $|z| = \sigma$ ($r_0 > \sigma$). We consider a circular element of $f(z)$, consisting of a neighborhood of z_0 (it must belong to the exterior of the circle) and the sum of the power series representing $f(z)$ in this neighborhood. We connect z_0 with the origin of coordinates by a segment L_θ of the ray $\operatorname{Arg} z = \theta$. It may happen that the element is continuable along L_θ; otherwise there will be a (first) singular point ζ_θ on L_θ in the continuation along this ray. Obviously, it depends only on θ and not on the position of z_0 on the ray (i.e. it does not depend on $r_0 > \sigma$) and the neighborhood of point z_0. It is then natural to call point ζ_θ a singular point for the continuation of $f(z)$ outside the circle $|z| = \sigma$ along the given ray. In all cases $0 \leqslant |\zeta_\theta| \leqslant \sigma$; the equality $|\zeta_\theta| = \sigma$ is valid if and only if $\sigma e^{i\theta}$ is a singular point of $f(z)$ on the circle $|z| = \sigma$. Hence, the set $\{\zeta_\theta\}$ is nonempty. Let us consider all its limit points. Each of these points, if it does not coincide with a point ζ_θ, first, contains in each of its neighborhoods some regular points of $f(z)$ in the continuations along rays and, second, cannot coincide with any of these regular points. Joining the set $\{\zeta_\theta\}$ with its limit points, we arrive at a closed set M, which we will call the *set of singularities* of $f(z)$ (in relation to continuations along all possible rays $\operatorname{Arg} z = \theta$). From the definition (and construction) it follows that M belongs to the closed circle $|z| \leqslant \sigma$ and can lie in no concentric circle of a smaller radius. It is obvious that for any φ the set M lies completely in the closed half-plane $\operatorname{Re}(ze^{-i\varphi}) \leqslant k(\varphi)$, which is the complement of the half-plane $\operatorname{Re}(ze^{-i\varphi}) > k(\varphi)$ with respect

to the entire plane. Whence M belongs to the intersection K of all the closed half-planes $\mathrm{Re}\,(ze^{-i\varphi}) \leqslant k(\varphi)$, $0 \leqslant \varphi \leqslant 2\pi$. From the definition of K it follows that this is a convex set (being an intersection of convex sets) and the smallest convex set containing M. The straight lines $\mathrm{Re}\,(ze^{-i\varphi}) = k(\varphi)$ are the *lines of support* of set K, and $k(\varphi)$ is the *support function* of K.

Now we have to find a way of calculating $k(\varphi)$ and build an apparatus for representing $f(z)$ in each of the half-planes $\mathrm{Re}\,(ze^{-i\varphi}) > k(\varphi)$. Obviously, it is advisable to use it only when $k(\varphi) < \sigma$ and, hence, when the half-plane contains points at which the given power series diverges.

The means for solving this problem are presented by entire functions of the exponential type; we will map the set of functions of the type

$$\sum_{0}^{\infty} \frac{c_n}{z^{n+1}} = f(z), \quad \overline{\lim} \sqrt[n]{|c_n|} = \sigma < +\infty \tag{9.7}$$

into the set of the entire functions of the exponential type. The set of functions (9.7) coincides with the set of all functions regular at the point at infinity and vanishing there. To each function $f(z)$ (9.7) there corresponds a function $F(z)$ defined by the series

$$F(z) = \sum_{0}^{\infty} \frac{c_n z^n}{n!}. \tag{9.8}$$

Since $\overline{\lim} \sqrt[n]{|c_n|} = \sigma < +\infty$, this is an entire function of the exponential type (see Sec. 7.12). Its order ρ is either less than unity (which is possible only if $\sigma = 0$) or exactly unity and then its type is σ. Conversely, each function of the exponential type can be represented in the form (9.8), where $\overline{\lim} \sqrt[n]{|c_n|} = \sigma < +\infty$, and, therefore, can have corresponding to it a function of the type (9.7).

The established one-to-one correspondence is known as the *Borel transformation* (1899), and the two functions $f(z)$ and $F(z)$ (the *Borel transform*) connected via the Borel transformation are said to be *associated in the sense of Borel*; $f(z)$ is regular at $z = \infty$ and is called the *lower function*, while $F(z)$ is a function of the exponential type and is called the *upper*.

Suppose that $h(\varphi)$ is the Phragmén-Lindelöf function for $F(z)$ (see Sec. 7.13). We wish to prove that the *lower function* $f(z)$ *admits a direct analytic continuation to the half-plane* $\mathrm{Re}\,(ze^{+i\varphi}) > h(\varphi)$. *We will also give for this continuation an integral representation in terms of Laplace's integral.*

Suppose that $F(z)$ is an entire function of the exponential type σ. For a fixed value of φ, $0 \leqslant \varphi \leqslant 2\pi$, we build the integral

$$I_\varphi(z) = \int_0^\infty e^{i\varphi} F(t)\, e^{-tz}\, dt, \tag{9.9}$$

where integration is carried out along the ray $t = \rho e^{i\varphi}$, $0 < \rho < +\infty$ (for $\varphi = 0$ we have Laplace's integral, or the Laplace transform, in its usual form; see Sec. 6.10).

Let us first show that $I_\varphi(z)$ is analytic in the half-plane $\operatorname{Re}(ze^{i\varphi}) > h(\varphi)$. We recall that $h(\varphi)$ satisfies the condition $-\infty < h(\varphi) \leqslant \sigma$ (Sec. 7.13). In view of the definition of $h(\varphi)$ we have, for any positive ε,

$$|F(\rho e^{i\varphi})| < e^{[h(\varphi)+\varepsilon]\rho} \text{ for } \rho > R(\varepsilon, \varphi).$$

Whence in each closed half-plane $\operatorname{Re}(ze^{i\varphi}) \geqslant h(\varphi) + 2\varepsilon$ we have, for $\rho > R(\varepsilon, \varphi)$,

$$|F(\rho e^{i\varphi}) \cdot e^{-\rho z e^{i\varphi}}| < e^{[h(\varphi)+\varepsilon]\rho} e^{-\rho[h(\varphi)+2\varepsilon]} = e^{-\rho\varepsilon},$$

from which follows the above proposition (the analyticity of $I_\varphi(z)$ is ensured by the fact that the modulus of the integrand is uniformly bounded in each half-plane $\operatorname{Re}(ze^{i\varphi}) \geqslant h(\varphi) + 2\varepsilon$ and that $I_\varphi(z)$ is absolutely convergent).

Now let us show that in the half-plane $\operatorname{Re}(ze^{i\varphi}) > \sigma$ (it belongs to the half-plane $\operatorname{Re}(ze^{i\varphi}) > h(\varphi)$ because $h(\varphi) \leqslant \sigma$) we have $I_\varphi(z) = f(z)$, i.e. the integral represents the lower function. Suppose that $\operatorname{Re}(ze^{i\varphi}) > \sigma + 2\varepsilon$; we choose a $T(\varepsilon)$ such that

$$|F(\rho e^{i\varphi})| < e^{[h(\varphi)+\varepsilon]\rho} \text{ for } \rho > T(\varepsilon) \text{ and } \int_{T(\varepsilon)}^\infty e^{-\varepsilon\rho}\, d\rho < \varepsilon.$$

After this we fix $T > T(\varepsilon)$ and choose a positive integer $N(\varepsilon)$ such that

$$|c_n| < (\sigma+\varepsilon)^n \text{ for } n > N(\varepsilon) \text{ and } \sum_{N(\varepsilon)}^\infty \frac{|c_n|}{n!} T^n < \varepsilon(\sigma + 2\varepsilon).$$

Next we fix a positive integer $N > N(\varepsilon)$ and represent $I_\varphi(z)$ in the following form:

$$I_\varphi(z) = \int_0^{\infty e^{i\varphi}} \sum_0^N \frac{c_n}{n!} t^n e^{-tz}\, dt + \int_0^{Te^{i\varphi}} \sum_{N+1}^\infty \frac{c_n}{n!} t^n e^{-tz}\, dt$$

$$+ \int_{Te^{i\varphi}}^\infty \sum_{N+1}^\infty \frac{c_n}{n!} t^n e^{-tz}\, dt.$$

Here

$$(\mathrm{I}) = \int_0^{\infty e^{i\varphi}} \sum_0^N \frac{c_n}{n!} t^n e^{tz}\, dt = \sum_0^N \frac{c_n}{n!} \int_0^{\infty e^{i\varphi}} t^n e^{-tz}\, dt = \sum_0^N \frac{c_n}{z^{n+1}};$$

the other two integrals (we denote them by (II) and (III), respectively) are estimated on the basis of the choice of T and N and the fact that

$$|t^n e^{-tz}| = \rho^n e^{-\rho \operatorname{Re}(ze^{i\varphi})} < \rho^n e^{-(\sigma+2\varepsilon)\rho} \quad \text{for} \quad \operatorname{Re}(ze^{i\varphi}) > \sigma + 2\varepsilon.$$

We have

$$|(\mathrm{II})| \leqslant \int_0^T \sum_{N+1}^\infty \frac{|c_n| T^n}{n!} e^{-(\sigma+2\varepsilon)\rho}\, d\rho < \varepsilon(\sigma+2\varepsilon) \int_0^\infty e^{-(\sigma+2\varepsilon)\rho}\, d\rho = \varepsilon,$$

$$|(\mathrm{III})| \leqslant \int_T^\infty \sum_{N+1}^\infty \frac{(\sigma+\varepsilon)^n \rho^n}{n!} e^{-(\sigma+2\varepsilon)\rho}\, d\rho$$

$$< \int_T^\infty \sum_0^\infty \frac{(\sigma+\varepsilon)^n \rho^n}{n!} e^{-(\sigma+2\varepsilon)\rho}\, d\rho = \int_T^\infty e^{-\rho\varepsilon}\, d\rho < \varepsilon.$$

Whence

$$\left| I_\varphi(z) - \sum_0^N \frac{c_n}{z^{n+1}} \right| < 2\varepsilon \quad \text{if} \quad \operatorname{Re}(ze^{i\varphi}) > \sigma + 2\delta \quad \text{and} \quad N > N(\varepsilon),$$

i.e.

$$I_\varphi(z) = \sum_0^\infty \frac{c_n}{z^{n+1}} = f(z) \quad \text{for} \quad \operatorname{Re}(ze^{i\varphi}) > \sigma.$$

But we have already established that $I_\varphi(z)$ is analytic in the half-plane $\operatorname{Re}(ze^{i\varphi}) > h(\varphi)$. For this reason, at $h(\varphi) < \sigma$ the function $I_\varphi(z)$ provides for a (direct) analytic continuation of $f(z)$ to the half-plane $\operatorname{Re}(ze^{i\varphi}) > h(\varphi)$. Consequently, the integral (9.9) is the means by which we can analytically continue the function $f(z)$, which we must in advance change to its Borel transform $F(z)$.

From the proved result it follows that

$$h(\varphi) \geqslant k(-\varphi),$$

since by definition $k(-\varphi)$ is the lower bound for those numbers c for which there exists a direct analytic continuation of $f(z)$ to the half-plane $\operatorname{Re}(ze^{i\varphi}) > c$.

Now we have to show that

$$h(\varphi) = k(-\varphi), \quad \text{i.e.} \quad k(\varphi) = h(-\varphi).$$

If we do this, the problem of calculating $k(\varphi)$ will be solved. More precisely, this problem will be reduced to finding the Phragmén-Lindelöf function for $F(z)$, the latter being associated with $f(z)$ in the sense of Borel.

To prove the above relationship, together with the convex set K (containing all the singular points of $f(z)$) we take a broader convex set K_ε whose support function is $k_\varepsilon(\varphi) = k(\varphi) + \varepsilon$. Since K is defined as the intersection of all closed half-planes $\operatorname{Re}(ze^{-i\varphi}) \leqslant \leqslant k(\varphi)$, $0 \leqslant \varphi \leqslant 2\pi$, the set K_ε is the intersection of the closed half-planes $\operatorname{Re}(ze^{-i\varphi}) \leqslant k(\varphi) + \varepsilon$. Therefore, we go over from K to K_ε by drawing circles with each point of the former as center and ε the radii and taking the union of all such circles. Since K belongs to the circle $|z| \leqslant \sigma$, the set K_ε is included in the circle $|z| \leqslant \sigma + \varepsilon$.

We denote the boundary of K by L and that of K_ε by L_ε. Next we consider the integral

$$\frac{1}{2\pi i} \oint_{L_\varepsilon} f(t)\, e^{tz}\, dt = F^*(z). \tag{9.10}$$

From the definition of $F^*(z)$ it follows that this function is analytic in the entire finite plane. In view of the analyticity of $f(z)$ in the domain $|z| > \sigma$ we conclude that

$$F^*(z) = \frac{1}{2\pi i} \oint_{|t|=\sigma+\varepsilon} f(t)\, e^{tz}\, dt = \frac{1}{2\pi i} \int_{|t|=\sigma+\varepsilon} \sum_0^\infty \frac{c_n}{t^{n+1}} e^{tz}\, dt$$

$$= \sum_0^\infty \frac{c_n}{2\pi i} \int_{|t|=\sigma+\varepsilon} \frac{e^{tz}}{t^{n+1}}\, dt = \sum_0^\infty \frac{c_n}{n!} z^n = F(z).$$

Hence, (9.10) does realize the Borel transformation by mapping the lower function $f(z)$ into the upper function $F(z)$. We note that Laplace's integral performs the inverse Borel transformation, mapping the upper function into the lower, although only in the half-plane $\operatorname{Re}(ze^{i\varphi}) > h(\varphi)$ for a given value of φ.

Formula (9.10) enables us to estimate $|F(z)|$ at points on the ray $z = re^{i\varphi}$, $0 < r < \infty$:

$$|F(re^{i\varphi})| \leqslant \frac{1}{2\pi} \int_{L_\varepsilon} |f(t)|\, |e^{r \operatorname{Re}(te^{i\varphi})}|\, |dt|$$

$$\leqslant e^{r \max\limits_{t \in L_\varepsilon} \operatorname{Re}(te^{i\varphi})} \, c_\varepsilon = c_\varepsilon e^{[k(-\varphi)+\varepsilon] r},$$

whence

$$h(\varphi) = \varlimsup_{r \to \infty} \frac{\ln |F(re^{i\varphi})|}{r} \leqslant k(-\varphi) + \varepsilon$$

and

$$h(\varphi) \leqslant k(-\varphi),$$

in view of the fact that ε is arbitrarily small.

Comparing the last inequality with the one we derived earlier, we have

$$h(\varphi) = k(-\varphi), \quad 0 \leqslant \varphi \leqslant 2\pi. \tag{9.11}$$

We have thus solved the problem of continuing analytically the function $f(z)$ in terms of its Borel transform $F(z)$, associated with $f(z)$ in the sense of Borel, the appropriate Laplace integral (9.9), and the Phragmén-Lindelöf function for $F(z)$.

The proved result also implies that the *Phragmén-Lindelöf function $h(\varphi)$ for an entire function $F(z)$ of the exponential type σ is the support function of a closed convex set I.* This set is symmetric with respect to the real axis to the smallest convex set K that contains all singular points of the function

$$f(z) = \sum_{0}^{\infty} \frac{a_n}{z^{n+1}} \quad (\overline{\lim} \sqrt[n]{|a_n|} = \sigma < +\infty)$$

in the event of an analytic continuation to the interior of the circle $|z| = \sigma$. Using the conjugation sign, we can write

$$I = \overline{K}.$$

The convex set I is called the *indicatrix* of $F(z)$, and K the *conjugate indicatrix*.

In Chap. 7 we proved that $h(\varphi)$ is continuous, whereby $k(\varphi)$ is continuous and

$$\max_{0 \leqslant \varphi \leqslant 2\pi} h(\varphi) = \max_{0 \leqslant \varphi \leqslant 2\pi} k(\varphi).$$

But this number is simply σ because on the circle $|z| = \sigma$ there is at least one singular point $\zeta = \sigma e^{i\varphi}$, and $k(\varphi) = \sigma$ for this value of φ. We have therefore proved that for a function $F(z)$ of the exponential type σ,

$$\max_{0 \leqslant \varphi \leqslant 2\pi} h(\varphi) = \sigma.$$

All these results were obtained by George Pólya. For illustration here are three examples.

Example 1. If we take

$$f(z) = \sum_{1}^{\infty} \frac{(-1)^{n-1}}{z^{2n}} = \frac{1}{1+z^2},$$

then

$$F(z) = \sum_{1}^{\infty} \frac{(-1)^{n-1} z^{2n-1}}{(2n-1)!} = \sin z.$$

The singular points of $f(z)$ are $\pm i$; their convex hull K is the section of the imaginary axis connecting these points. We then have: $k(\varphi) = |\sin \varphi|$, whence the Phragmén-Lindelöf function for $\sin z$ is $h(\varphi) = k(-\varphi) = |\sin \varphi|$. This result follows directly from Euler's formula:

$$\sin z = \sin(re^{i\varphi}) = \frac{e^{ire^{i\varphi}} - e^{-ire^{i\varphi}}}{2i}.$$

Indeed,

$$|e^{ire^{i\varphi}}| = e^{-r\sin\varphi} \quad \text{and} \quad |e^{-ire^{i\varphi}}| = e^{r\sin\varphi},$$

and, hence, for any value of φ, $0 < \varphi < \pi$ or $\pi < \varphi < 2\pi$, the principal term in $\ln|\sin(re^{i\varphi})|$ is $r|\sin\varphi|$, whence

$$h(\varphi) = \overline{\lim_{r\to\infty}} \frac{\ln|\sin(re^{i\varphi})|}{r} = |\sin\varphi|.$$

This is also true, obviously, for $\varphi = 0$ and $\varphi = \pi$.

Example 2. Let us take $f(z) = \sum_{0}^{\infty} \frac{c_n}{z^{n+1}} \not\equiv 0$, with $\overline{\lim} \sqrt[n]{|c_n|} = \sigma = 0$. Then K coincides with the one and only singular point of $f(z)$, the origin of coordinates, and hence $k(\varphi) = 0$. For the Borel transform $F(z) = \sum_{0}^{\infty} \frac{c_n}{n!} z^n$ we then have

$$h(\varphi) = k(-\varphi) \equiv 0.$$

Example 3. Let us take $f(z) = \sum_{0}^{\infty} \frac{c_n}{z^{n+1}}$, with $c_n \geqslant 0$ and $\overline{\lim} \sqrt[n]{c_n} = \sigma < +\infty$. We wish to show that $z = \sigma$ is the singular point of $f(z)$. If this is true, we have arrived again at Pringsheim's theorem (see Sec. 9.7). It suffices to prove that $k(0) = h(0) = \sigma$. But $\max_{0\leqslant\varphi\leqslant 2\pi} h(\varphi)$ is σ (see above); on the other hand, the maximum of $|F(re^{i\varphi})|$, $0 \leqslant \varphi \leqslant 2\pi$, is attained at $\varphi = 0$ because

$$|F(re^{i\varphi})| = \left| \sum_{0}^{\infty} \frac{c_n}{n!} r^n e^{in\varphi} \right| \leqslant \sum_{0}^{\infty} \frac{c_n}{n!} r^n = F(r).$$

10

MAPPINGS PERFORMED BY ANALYTIC FUNCTIONS. RIEMANN'S MAPPING THEOREM. THE SCHWARZ-CHRISTOFFEL TRANSFORMATION FORMULA

10.1

MAPPING A DOMAIN BY AN ANALYTIC FUNCTION

Let $w = f(z) \not\equiv$ const be single-valued and analytic in a domain G. We wish to show that the set D of all its values attained in G is a domain, too. To this end we must show that each point $w_0 = f(z_0)$ belongs to D together with a neighborhood and that every two points in D can be linked by a continuous curve lying entirely in D. We build a closed circle C_0: $|z - z_0| \leqslant \rho$ such that it belongs to G and so that a value w_0 of this function corresponds only to z_0 and no other point in C_0. The last requirement can be met if we take ρ small. If we were to assume the contrary, then in each neighborhood of z_0 there would be distinct points at which $f(z)$ would admit the same value w_0, which is impossible by the uniqueness theorem ($f(z) \not\equiv w_0$). We introduce the notation $\mu = \min\limits_{|z-z_0|=\rho} |f(z) - w_0|$; obviously, $\mu > 0$ in view of the fact that $f(z)$ is continuous and is not equal to w_0 on the circle $|z - z_0| = \rho$. We wish to prove that the circle K_0: $|w - w_0| < \mu$ belongs to D. Indeed, the equation $f(z) - w_0 = 0$ has at least one root inside C_0, namely $z = z_0$ (the multiplicity of the root must also be taken into account). Suppose that $w' \in K_0$ $(w' = w_0)$; since $|w_0 - w'| < \mu$ and $|f(z) - w_0| \geqslant \mu$ on the circle $|z - z_0| = \rho$, according to Rouche's theorem the equation $[f(z) - w_0] + (w_0 - w') = f(z) - w' = 0$ has inside this circle as many roots as the equation $f(z) - w_0 = 0$, i.e. at least one. We denote one of the roots by z'; this means that $f(z') = w'$, i.e. each point w' in K_0 is a value of $f(z)$ attained inside C_0. Therefore, K_0 belongs to D.

Let us assume that $w_1 = f(z_1)$ and $w_2 = f(z_2)$ are any two points in D. Since z_1 and z_2 belong to G, these points can be connected by a curve γ: $z = \lambda(t)$, $\alpha \leqslant t \leqslant \beta$, lying in the interior of G ($\lambda(\alpha) = z_1$, $\lambda(\beta) = z_2$). Obviously, $w = f(z) = f[\lambda(t)]$ $(\alpha \leqslant t \leqslant \beta)$ is also a continuous curve lying in D and connecting w_1 with w_2 ($w_1 = f(z_1) = f[\lambda(\alpha)]$ and $w_2 = f(z_2) = f[\lambda(\beta)]$).

We have proved the following

Theorem. *The set D of values admitted in a domain G by a single-valued analytic function* $w = f(z) \not\equiv \text{const}$ *is a domain. In other words, an analytic function that is not a constant always maps a domain into a domain.*

The theorem remains valid for the case where the domain G of the extended plane contains point ∞ and/or $f(z)$ has poles in G. To prove this statement we need only to introduce auxiliary linear-fractional transformations. Specifically, if $z_0 = \infty$, we proceed with the mapping $z' = 1/z$, which maps a neighborhood of the point at infinity to that of the origin of coordinates; if $w_0 = f(z_0) = \infty$ (z_0 is either a finite point or ∞, a pole of $f(z)$), we introduce the auxiliary mapping $w' = 1/f(z)$, which maps the function $f(z)$ with a pole at z_0 into the function $1/f(z)$ with a zero at the same point. These transformations, obviously, reduce the case to the previous one, where the values of both the independent variable and the function are finite. Similar remarks can be made in relation to the statements in the next sections.

10.2
THE MAXIMUM MODULUS PRINCIPLE AND SCHWARZ'S LEMMA

The theorem we have just proved provides for a proof of the *maximum modulus principle* that differs from that given in Sec. 6.11. We start by restating this principle:

Suppose that $w = f(z) \not\equiv \text{const}$ *is single-valued and analytic at all points of a domain G in the extended plane. Then there is no point* $z_0 \in G$ *at which* $|f(z)|$ *has a maximum. In other words, if we know that the modulus of a function* $f(z)$ *single-valued and analytic in G has a maximum at a point in G, then* $f(z) \equiv \text{const}$.

The proof (in the first formulation) follows from the fact that point $w_0 = f(z_0)$ belongs to the set of values of $f(z)$ together with a neighborhood K_0: $|w - w_0| < \mu$ (we preserved the notations of Sec. 10.1). From this neighborhood we select a point w' such that $|w'| > |w_0|$; by the theorem proved in the previous section $w' = f(z')$, where z' lies inside a neighborhood C_0: $|z - z_0| \leqslant \rho$ of point z_0. Hence, in a neighborhood of z_0 (this neighborhood can be chosen arbitrarily small by taking ρ small) there will always be a point z' for which $|f(z')| > |f(z_0)|$. But this means that $|f(z)|$ cannot attain a maximum at z_0. The reader can easily prove along similar lines that when $f(z)$ does not vanish in G, its modulus can have no minimum in G.

As an application of the maximum modulus principle, which we will use subsequently, we will prove

Schwarz's lemma (1869). *If $f(z)$ is a function that is single-valued and analytic in the unit circle and satisfies the conditions that $f(0) = 0$ and $|f(z)| \leqslant 1$ $(|z| < 1)$, then it also satisfies the conditions $|f'(0)| \leqslant 1$ and $|f(z)| \leqslant |z|$ $(|z| < 1)$. The equality $|f'(0)| = 1$ or $|f(z_0)| = |z_0|$ (at least at one point z_0 such that $z_0 \neq 0$ and $|z_0| < 1$) takes place only when $f(z)$ is a linear function of the type $e^{i\alpha}z$ (α real and constant).*

Proof. Let us assume that $f(z) = c_1 z + c_2 z^2 + \ldots$ $(|z| < 1)$; we put $\varphi(z) = f(z)/z = c_1 + c_2 z + \ldots$. Obviously, $\varphi(z)$ is a function that is analytic in the unit circle and satisfies the condition $\varphi(0) = c_1 = f'(0)$.

We take the value of $\varphi(z)$ at some point z' of the unit circle. If r is such that $|z'| < r < 1$, then by the above-proved,

$$|\varphi(z')| \leqslant \max_{|z|=r} |\varphi(z)|.$$

But $\max\limits_{|z|=r} |\varphi(z)| = \max\limits_{|z|=r} |f(z)/z| \leqslant 1/r$ because $|f(z)| \leqslant 1$; whence $|\varphi(z')| \leqslant 1/r$, or, fixing z' and tending r to unity, we have

$$|\varphi(z')| \leqslant 1.$$

In particular, $|\varphi(0)| = |f'(0)| \leqslant 1$ at $z' = 0$ and $|\varphi(z_0)| = |f(z_0)/z_0| \leqslant 1$, i.e. $|f(z_0)| \leqslant |z_0|$, at $z' = z_0 \neq 0$. The equality sign in any of these two relationships would mean that, at a point z' of the unit circle, $|\varphi(z)|$ attains a maximum equal to unity; this is possible only in the case where $\varphi(z) \equiv \text{const} = e^{i\alpha}$ (since $|\varphi(z)| = 1$), i.e. $f(z) = e^{i\alpha}z$.

Let us apply the maximum modulus principle to harmonic functions. We wish to prove that a *function $u(x, y) \not\equiv$ const that is single-valued and harmonic in a domain G can have neither maxima nor minima in any point in G*. Indeed, let (x_0, y_0) be a point in G and let U_0 be a neighborhood of this point belonging to G. We build a function $v(x, y)$ that is harmonic in G and conjugate to $u(x, y)$ (see Sec. 2.13). Then $u(x, y) + iv(x, y) = f(z)$ is single-valued and analytic in U_0, and so are the functions $F_1(z) = e^{f(z)}$ and $F_2(z) = e^{-f(z)}$ in the same neighborhood. Since they are not constants, we can apply to each the maximum modulus principle. For this reason $|F_1(z_0)| = e^{u(x_0, y_0)}$ and $|F_2(z_0)| = e^{-u(x_0, y_0)}$ cannot be maximal, and the value $u(x_0, y_0)$ of the harmonic function can be neither maximal nor minimal.

If $u(x, y)$ is a function continuous in a closed bounded domain $\overline{G}$ and harmonic in G, by the above-proved its maximal or minimal value is attained only at the boundary points of the domain. In particular, if $u(x, y)$ remains constant on the boundary of G, then $u(x, y) \equiv$ const in G. For this reason two functions $u_1(x, y)$ and $u_2(x, y)$ that are continuous in a closed domain $\overline{G}$, harmonic in this domain,

and attain the same values at the boundary points of G, must coincide everywhere in G. This means that the Dirichlet problem, which consists of finding a function continuous in $\bar{G}$ and harmonic in G by its preassigned values on the boundary of G, can have only one solution.

10.3

LOCAL CRITERION FOR UNIVALENCE

Let us study in more detail the mapping $w = f(z)$ in a closed neighborhood C_0: $|z - z_0| \leqslant \rho$. We assume, first, that $f'(z_0) = 0, \ldots, f^{(p-1)}(z_0) = 0, f^{(p)}(z_0) \neq 0$ $(p \geqslant 2)$. The derivative $f'(z)$ may vanish not only at point z_0 but at other points in C_0; since z_0 is not a limit point for the set of zeros of $f'(z)$ (otherwise we would have $f'(z) \equiv 0$ and $f(z) \equiv \text{const}$), we can choose ρ so small that $f'(z)$ does not vanish at a single point of C_0 that differs from z_0. Suppose that we have chosen ρ. Then, preserving the conditions and notations of Sec. 10.1, we may say that for every $w' \neq w_0$ and $|w' - w_0| < \mu$ the equation $f(z) - w' = 0$ has as many roots inside C_0 as the equation $f(z) - w_0 = 0$. But the latter has p roots (z_0 is a p-order root); whence $f(z) - w' = 0$ has p roots inside C_0, too. Not one of these roots z' can be multiple since $z' \neq z_0$ (otherwise we would have $w' = f(z') = f(z_0) = w_0$, which is false) and, hence, $f'(z') \neq 0$. Thus, the equation $f(z) - w' = 0$ has p distinct roots $z_1, z_2, \ldots, z_p$ inside C_0, i.e. there are p distinct points in this neighborhood at which $f(z)$ admits the same value w'. This implies that a function whose derivative vanishes at least at one point in a domain cannot be univalent in this domain. In other words, *if $f(z)$ is univalent in a domain G, its derivative can vanish in no point in G.*

The condition that $f'(z) \neq 0$ $(z \in G)$ is necessary but not sufficient for univalence, as demonstrated by the function e^z. Its derivative $(e^z)' = e^z$ vanishes nowhere, and yet e^z admits equal values at all points of the fork $z + 2n\pi i$ $(n = 0, \pm 1, \pm 2, \ldots)$.

However, *for the point z_0 at which $f'(z_0) \neq 0$ we can always build a closed neighborhood C_0: $|z - z_0| \leqslant \rho_0$ such that $f(z)$ is univalent in it.* If $f(z) = \sum_0^\infty a_n (z - z_0)^n$ is the expansion of $f(z)$ in a neighborhood $|z - z_0| < r$ of point z_0 $(a_1 = f'(z_0) \neq 0)$, then the series $f'(z) = \sum_1^\infty n a_n (z - z_0)^{n-1}$ is convergent in the same neighborhood; hence, for $\rho < r$ the series $\sum_2^\infty n |a_n| \rho^{n-1}$ converges and its sum tends to zero as $\rho \to 0$. We choose ρ_0 $(0 < \rho_0 < r)$

so that $\sum_{2}^{\infty} n\,|a_n|\,\rho_0^{n-1} < |a_1|$. Then for any two points z_1 and z_2 $(z_1 \neq z_2)$ in the closed circle $|z-z_0| \leqslant \rho_0$ we have

$$|f(z_1)-f(z_2)| = \left|\sum_{0}^{\infty} a_n\,[(z_1-z_0)^n-(z_2-z_0)^n]\right|$$

$$= \left|a_1(z_1-z_2)+\sum_{2}^{\infty} a_n\,[(z_1-z_0)^{n-1}+\ldots+(z_2-z_0)^{n-1}]\,(z_1-z_2)\right|$$

$$\geqslant |z_1-z_2|\left\{|a_1|-\sum_{2}^{\infty}|a_n|\,[|z_1-z_0|^{n-1}+\ldots+|z_2-z_0|^{n-1}]\right\}$$

$$\geqslant |z_1-z_2|\left\{|a_1|-\sum_{2}^{\infty} n\,|a_n|\,\rho_0^{n-1}\right\}>0, \quad \text{i.e.} \quad f(z_1)\neq f(z_2),$$

which proves the univalence of $f(z)$ in the closed circle $|z - z_0| \leqslant \rho_0$.

10.4

INVERSION OF ANALYTIC FUNCTIONS

Let $w = f(z)$ be a single-valued analytic function that is univalent in a domain G without the point at infinity. As we have proved in Sec. 10.1, *this function maps G into a domain D. We wish to show that the inverse function $z = \varphi(w)$ is also single-valued, analytic, and univalent in D.* The fact that $\varphi(w)$ is single-valued follows directly from the univalence of $f(z)$. Indeed, if $z_1 \neq z_2$ are two values of $\varphi(w)$ at point $w_0 \in D$, we must have $f(z_1) = f(z_2) = w_0$, which contradicts the fact that $f(z)$ is univalent. In a similar manner, the univalence of $\varphi(w)$ follows from the fact that $f(z)$ is single-valued; if we assume that $\varphi(w_1) = \varphi(w_2) = z_0$, where $w_1 \neq w_2$, then we must have $f(z_0) = w_1$ and $f(z_0) = w_2$, which contradicts the fact that $f(z)$ is single-valued. Let us prove the continuity of $\varphi(w)$. Suppose that $w_0 \in D$ and $\varphi(w_0) = z_0$. For z_0 we build a closed neighborhood C_0: $|z - z_0| \leqslant \rho$, which was used in Sec. 10.1. Then each point w' that belongs to the circle K_0: $|w - w_0| < \mu$, with $\mu = \min\limits_{|z-z_0|=\rho} |f(z) - w_0| > 0$, corresponds to a value of $f(z)$ at a point z' inside C_0; in other words, $z' = \varphi(w')$ lies inside C_0 if $w' \in K_0$. For an arbitrary positive number ε we choose $\rho < \varepsilon$; then for the respective $\mu = \mu(\rho)$ we have:

$$\text{if} \quad |w' - w_0| < \mu, \quad \text{then} \quad |\varphi(w') - \varphi(w_0)| < \rho < \varepsilon.$$

The continuity of $\varphi(w)$ in D is proved. We still have to prove that $\varphi(w)$ has a derivative at each point $w_0 \in D$. Suppose that $\varphi(w_0) = z_0$; then for $w \neq w_0$ we have $\varphi(w) = z \to \varphi(w_0) = z_0$ as $w \to w_0$

$(z \neq z_0)$ and, hence,

$$\lim_{w \to w_0} \frac{\varphi(w) - \varphi(w_0)}{w - w_0} = \lim_{z \to z_0} \frac{z - z_0}{f(z) - f(z_0)} = \frac{1}{f'(z_0)}$$

$(f'(z_0) \neq 0$ in view of the univalence of $f(z))$. The proof of the theorem is complete.

Let us apply this result to *local inversion* of an arbitrary single-valued (not univalent, generally speaking) and analytic function $w = f(z)$ in a neighborhood of the point $w_0 = f(z_0)$ $(z_0 \in G)$.

We first assume that $f'(z_0) \neq 0$; then according to Sec. 10.3 there is a circle C_0: $|z - z_0| < \rho_0$ in which $f(z)$ is univalent. The

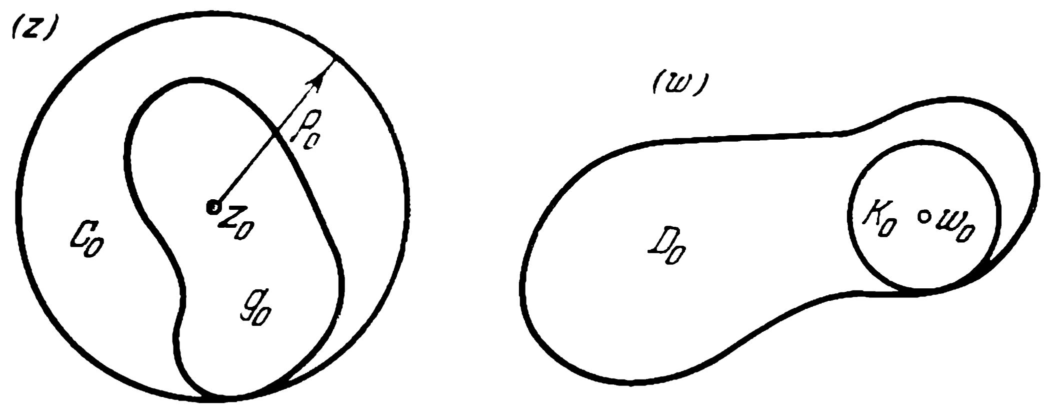


Fig. 71

function $w = f(z)$ maps this circle conformally and in a one-to-one manner into a domain D_0 in the w plane that contains point w_0.

By the above-proved theorem, the inverse function $z = \varphi(w)$ is single-valued, analytic, and univalent in D_0—it maps the circle K_0: $|w - w_0| < r_0$ (r_0 is the distance from point w_0 to the boundary of D_0) in a one-to-one manner and conformally into a domain g_0 in the z plane that contains point z_0 (Fig. 71). In K_0 the function $\varphi(w)$ can be expanded in a power series:

$$z = \varphi(w) = c_0 + c_1(w - w_0) + \ldots + c_n(w - w_0)^n + \ldots \quad (10.1)$$

$$(|w - w_0| < r_0).$$

The expansion coefficients c_n here can be calculated by employing the method of undetermined coefficients, for example. (For this we must substitute for $w - w_0 = f(z) - f(z_0)$ the appropriate power series, whose expansion coefficients are assumed to be known: $f(z) - f(z_0) = a_1(z - z_0) + a_2(z - z_0)^2 + \ldots$.) We have, in particular,

$$c_0 = z_0 \quad \text{and} \quad c_1 = \frac{1}{a_1}.$$

Now suppose that $f'(z_0) = \ldots = f^{(p-1)}(z_0) = 0$ and $f^{(p)}(z_0) \neq$ $\neq 0$ $(p \geqslant 2)$. We represent the expansion

$$f(z) = a_0 + a_p (z - z_0)^p + \ldots \left(a_0 = f(z_0) = w_0, \quad a_p = \frac{f^{(p)}(z_0)}{p!} \neq 0\right), \tag{10.2}$$

which is convergent in the circle $|z - z_0| < \rho$, in the form

$$f(z) = a_0 + a_p (z - z_0)^p [1 + \alpha(z)],$$

where $\alpha(z) = \frac{a_{p+1}}{a_p}(z - z_0) + \frac{a_{p+2}}{a_p}(z - z_0)^2 + \ldots$ is a function analytic in the circle $|z - z_0| < \rho$. Suppose that ρ_1 $(0 < \rho_1 \leqslant \rho)$ is such that $|\alpha(z)| < 1$ in $|z - z_0| < \rho_1$; then $\beta(z) = [1 + \alpha(z)]^{1/p} =$ $= 1 + \sum\limits_{n=1}^{\infty} \binom{1/p}{n} [\alpha(z)]^n$ is single-valued and analytic in this circle and satisfies the condition that $\beta(z_0) = 1$. Whence $\gamma(z) = (z - z_0)\beta(z)$ is also single-valued and analytic in $|z - z_0| < \rho_1$ and such that $\gamma'(z_0) = 1 \neq 0$. Obviously, $f(z)$ can be expressed in terms of $\gamma(z)$:

$$w = f(z) = a_0 + a_p [\gamma(z)]^p. \tag{10.3}$$

Consider the transformation

$$\zeta = \gamma(z) = (z - z_0) + b_2 (z - z_0)^2 + \ldots. \tag{10.4}$$

To this we can apply the results obtained earlier for the transformation $w = f(z)$ in the case where $f'(z_0) \neq 0$. Therefore, there is a domain g_0 in $|z - z_0| < \rho$ that contains point z_0, which point is mapped by (10.4) in a one-to-one manner and conformally into a circle K_0: $|\zeta| < r_1$. The function $z = \delta(\zeta)$, which is the inverse of (10.4), has in this circle the expansion

$$z = \delta(\zeta) = \zeta + c_2 \zeta^2 + \ldots. \tag{10.5}$$

To obtain the transformation (10.2) it suffices to perform the mapping (10.4) and then subject K_0 to the mapping

$$w = a_0 + a_p \zeta^p \quad \left(a_0 = w_0, \; a_p = \frac{f^{(p)}(z_0)}{p!} \neq 0\right). \tag{10.6}$$

This last mapping is not one-to-one because the p different points in K_0, say ζ, $\zeta e^{2\pi i/p}$, $\ldots$, $\zeta e^{2\pi i(p-1)/p}$ $(\zeta \neq 0)$, are mapped into one point $w = a_0 + a_p \zeta^p$. But if we present the image of K_0 as a p-sheeted Riemann surface, a p-sheeted circle centered at $a_0 = w_0$, the one-to-one manner of the mapping is restored. We can build this p-sheeted circle as follows. We divide K_0 into $2p$ equal sectors S_j: $(j - 1)\pi/p < \alpha < j\pi/p$, $0 < r < r_1$ $(j = 1, 2, \ldots, 2p)$ and subject each to the mapping (10.6). As a result we obtain $2p$ copies of the upper and lower semicircles D_j: $(j - 1)\pi < \varphi < j\pi$, $0 < r <$

$< r_1^p$ (φ and r are polar coordinates with the origin at w_0). Now we only have to glue each semicircle D_j with D_{j+1} ($j = 1, 2, \ldots, 2p - 1$) along the common radius $\varphi = j\pi$ ($0 < r < r_1^p$); the last semicircle D_{2p} is glued to the first semicircle D_1 along the common radius $\varphi = 0$ ($0 < r < r_1^p$). From (10.5) and (10.6) we immediately obtain the inverse $z = \varphi(w)$ of the function $w = f(z)$ in a neighborhood of point w_0; namely, (10.6) yields

$$\zeta = \left(\frac{1}{a_p}\right)^{1/p} (w - w_0)^{1/p}$$

(here to the coefficient $(1/a_p)^{1/p}$ we may ascribe one fixed value of the pth root of $1/a_p$; the various values of ζ corresponding to the same value of w can be obtained as a result of revolutions about point w_0). Substituting into (10.5), we have

$$z = \varphi(w) = \sum_{1}^{\infty} A_n (w - w_0)^{n/p} \quad \left(A_n = c_n \left(\frac{1}{a_p}\right)^{n/p}, \; c_1 = 1\right). \quad (10.7)$$

This is an expansion that converges at $|w - w_0| < r_1^p$. It shows that the function $z = \varphi(w)$ has an algebraic branch point of multiplicity p at w_0. The p-sheeted circle that we have built constitutes the part of the Riemann surface for $\varphi(w)$ in neighborhood of w_0 and is mapped by (10.7) into a one-sheeted domain g_0 in the z plane.

We have therefore demonstrated that *when a single-valued analytic function $w = f(z)$ maps a domain G, for each point $z_0 \in G$ we can select a domain g_0, which contains point z_0 and belongs to G, that the function $f(z)$ maps in a one-to-one manner into a one-sheeted (if $f'(z_0) \neq 0$) or many-sheeted (if $f'(z_0) = 0$) circle centered at $w_0 = f(z_0)$.* If for each point z we choose a domain g, we can glue the circles $\{K\}$ together, connecting two circles K_1 and K_2 if and only if they overlap and if the same is true for their preimages g_1 and g_2. The reader can easily see that this process is similar to that of building the Riemann surface for an analytic function by its elements (see Chap. 9).

In the case at hand we are dealing with the elements of $z = \varphi(w)$ defined by expansions of type (10.1) (in a one-sheeted circle) or (10.7) (in a p-sheeted circle; $p \geqslant 2$), although the second case does not comply with the definition of a function element given in Chap. 9 because the sum of (10.7) has a singular point at $w = w_0$ (a branch point). To bring the manner of construction presented here closer to that in Chap. 9, it would be sufficient to "dismantle" the p-sheeted circle by, say, dissecting it into semicircles D_j and taking in each semicircle the corresponding branch of the analytic function (10.7) (namely, in the semicircle $(j - 1)\pi < \varphi < j\pi$, $0 < r < r_1$, we

take the unique sum of the series $\sum_{1}^{\infty} A_n r^{n/p} \left(\cos \frac{n\varphi}{p} + i \sin \frac{n\varphi}{p}\right)$; the values on this branch are depicted by points of the part of g_0 into which (10.5) maps the circular sector S_j: $(j-1)\pi/p < \alpha < j\pi/p$, $0 < r < r_1$). More often, however, the notion of an analytic function element is generalized to include p-sheeted circles in the family of domains of elements, and functions with algebraic branch points of multiplicity p at the centers of the circles in the family of functions defined in such domains.

10.5

EXTENDING THE CONCEPT OF UNIVALENCE TO FUNCTIONS WITH POLES

Cases where either z_0 or $w_0 = f(z_0)$ becomes infinite can be reduced to those examined above, as we have already noted at the end of Sec. 10.1. Here are some results given without proof.

If $z_0 = \infty$ and $f(z)$ is regular at this point, in a neighborhood of the point at infinity we have

$$f(z) = A_0 + \frac{A_1}{z} + \frac{A_2}{z^2} + \frac{A_3}{z^3} + \ldots .$$

It is easy to see that in this case all the derivatives vanish: $f'(\infty) = f''(\infty) = \ldots = 0$; the nature of the mapping, however, depends on the coefficients $A_1, A_2, \ldots$ rather than on the derivatives. If $A_1 = A_2 = \ldots = A_{p-1} = 0$ but $A_p \neq 0$ $(p \neq 0)$, in any neighborhood of the point at infinity there are p distinct points at which $f(z)$ admits the same value (cf. Sec. 10.3). Therefore, *for a function $f(z)$ to be univalent in a domain G with the point at infinity it is necessary that $A_1 = -\operatorname{Res}_{z=\infty} f(z) \neq 0$. If $f(z)$ is regular at point $z = \infty$ and $A_1 \neq 0$, then this is sufficient for $f(z)$ to be univalent in a neighborhood of point ∞ (Sec. 10.3).*

Let us expand the idea of a univalent function by assuming that a single-valued analytic function $f(z)$ has poles in a domain G. From the fact that this function is univalent ($f(z_1) \neq f(z_2)$ if $z_1 \neq z_2$) it follows that it can have only one pole. We wish to show that this pole must be simple. Suppose that $z_0 \in G$ is the pole of $f(z)$; for the sake of definiteness we assume that $z_0 \neq \infty$. Then

$$f(z) = \frac{A_{-p}}{(z-z_0)^p} + \ldots + \frac{A_{-1}}{z-z_0} + A_0 + \ldots \quad (p \geqslant 1,\ A_{-p} \neq 0)$$

in a neighborhood of z_0. If $f(z)$ is univalent in the neighborhood, $1/f(z) = \alpha_p (z-z_0)^p + \alpha_{p+1}(z-z_0)^{p+1} + \ldots$ $(\alpha_p = 1/A_{-p} \neq 0)$ is univalent in the same neighborhood. But in Sec. 10.3 we proved

that this is possible only if p is unity, i.e. when the order of the pole of $f(z)$ is unity. In our generalization of the idea of univalence the theorem proved at the beginning of Sec. 10.4 remains valid for domains containing the point at infinity if we add the statement that the inverse $z = \varphi(w)$ of the univalent function may have a simple pole in D.

We note, finally, that a *one-to-one mapping of G into D realized by a univalent function $w = f(z)$ is conformal (of the first kind) at all points of G*. If $z_0 \neq \infty$ and $f(z_0) \neq \infty$, the above statement follows from the fact that $f'(z_0) \neq 0$. Suppose that $z_0 \neq \infty$ but $f(z_0) = \infty$; then $w = f(z) = A_{-1}/(z - z_0) + A_0 + \ldots$ $(A_{-1} \neq 0)$ and $w' = 1/w = 1/f(z) = a_1(z - z_0) + \ldots$, where $a_1 = 1/A_{-1} = \left.\frac{dw'}{dz}\right|_{z=z_0} \neq 0$. Since the mapping $w' = 1/f(z)$ is conformal at point z_0, so is the mapping $w = f(z)$. The case with $z_0 = \infty$ can be reduced to the previous one by introducing the mapping $z' = 1/z$ (here either $f(\infty) \neq \infty$, then $w = f(1/z') = A_0 + A_1 z' + A_2 z'^2 + \ldots$ $(A_1 \neq 0)$, or $f(\infty) = \infty$, then $w = f(1/z') = \frac{A_{-1}}{z'} + A_0 + A_1 z' + \ldots$ $(A_{-1} \neq 0)$).

10.6 RIEMANN'S MAPPING THEOREM. THE UNIQUENESS OF A MAPPING

Let us consider several general propositions dealing with conformal mapping theory. We have already seen that every function $w = f(z)$ that is univalent in a domain G realizes a one-to-one and conformal mapping (of the first kind) of this domain into another domain, D.*

In 1926 D. E. Men'shov proved the converse theorem: *each one-to-one and conformal mapping (of the first kind) of one domain G in the extended plane into another domain is realized by a function univalent in G*. For this reason, the expressions "conformal mapping (of the first kind) of a domain" and "mapping of a domain by a univalent function" are considered equivalent.

The following theorem lies at the base of conformal mapping theory:

Riemann's mapping theorem (1851). *A simply connected domain G whose boundary consists of more than one point can be conformally mapped into a circle (for example, the unit circle) and in an infinite number of ways.*

The condition imposed on G is significant. Indeed, let the boundary of G consist of only one point ζ. If $\zeta \neq \infty$, we can map it into ∞ by the conformal mapping $z' = 1/(z - \zeta)$. Suppose that we have already realized this mapping process; then G coincides

* Here and in what follows we assume that the function is single-valued and analytic in G everywhere except, perhaps, at one simple pole.

with the finite plane. The function $w = f(z)$ that maps G conformally into the unit circle $|w| < 1$ must be analytic in the entire finite plane, i.e. be an entire function, and bounded: $|f(z)| < 1$. But by Liouville's theorem (see Sec. 6.2) such a function is a constant, i.e. maps the entire plane into one point. This contradiction proves that it is impossible to map a domain with a single boundary point into a circle conformally.

When the hypothesis is true, the conformal mapping can be achieved in an infinite number of ways. Specifically, the function $w = f(z)$ that conformally maps G into the unit circle can be subjected to various restrictions. For instance, we can require that a fixed point $z_0 \in G$ be mapped into the circle's center ($f(z_0) = 0$) and, in addition, that tangents to curves at this point preserve their direction (this means that $\arg f'(z_0) = 0$, i.e. $f'(z_0)$ is a positive real number). Indeed, suppose that $w_1 = f_1(z)$ is a function that conformally maps G into the circle $|w_1| < 1$. Point z_0 is mapped in the process into $f_1(z_0)$ and tangents to curves at z_0 turn through an angle equal to $\arg f_1'(z_0) = \alpha$. If $f_1(z_0)$ is not zero, we map the circle conformally into itself in a way such that $f_1(z_0)$ is transformed into the center. This can be done by applying the mapping $w_2 = \frac{w_1 - f_1(z_0)}{1 - \overline{f_1(z_0)}\, w_1}$ (see Sec. 3.9). Obviously, the function $w_2 = f_2(z) = \frac{f_1(z) - f_1(z_0)}{1 - \overline{f_1(z_0)}\, f_1(z)}$ conformally maps G into the circle $|w_2| < 1$ so that point z_0 becomes the circle's center. Also, $f_2'(z_0) = \frac{f_1'(z_0)}{1 - |f_1(z_0)|^2}$ and, hence, $\arg f_2'(z_0) = \arg f_1'(z_0) = \alpha$.

Last, we must put $w = f(z) = e^{-i\alpha} f_2(z)$ to obtain a conformal mapping satisfying the restrictions.

Let us turn to the proof of Riemann's mapping theorem. We start by demonstrating that among the functions that are univalent in a given domain (linear-fractional functions are examples of such functions) there are functions whose moduli are bounded.

When G is bounded, a function of this kind is $f(z) = z$.

When G is not bounded but there is a point z_0 exterior to G, there is a circle $|z - z_0| < \rho$ lying in the exterior of G. Then the function $f(z) = 1/(z - z_0)$ is an example of a function univalent in G and bounded ($|f(z)| < 1/\rho$ at $z \in G$).

Finally, suppose that G is not bounded and has no exterior points. According to the hypothesis there are at least two boundary points for G, say α and β. In view of the fact that G is simply connected, the two points belong to a continuum Γ that is the boundary of the domain. Let us take the function $F(z) = \sqrt{\frac{z-\alpha}{z-\beta}}$. This is a two-to-one function with two branch points: α and β. Since G can be obtained from the entire plane by deleting a continuum connect-

ing points α and β, the function $F(z)$ is separated in G into two branches, $F_1(z)$ and $F_2(z)$, whose values at each point in G differ only in sign. These two functions are univalent in G since

$$\sqrt{\frac{z_1-\alpha}{z_1-\beta}}=\sqrt{\frac{z_2-\alpha}{z_2-\beta}}$$

yields

$$\frac{z_1-\alpha}{z_1-\beta}=\frac{z_2-\alpha}{z_2-\beta}$$

and, hence,

$$z_1 = z_2.$$

Whence the functions $w = F_1(z)$ and $w = F_2(z)$ conformally map G into two domains: G_1 and G_2. Obviously, G_1 and G_2 have no common points because the presence of such a point $w = F(z_1) = F(z_2)$ would mean, according to the above reasoning, that $z_1 = z_2 = z$, which contradicts the fact that $F_1(z) = -F_2(z)$ and neither function vanishes or becomes infinite. Consequently, each point w_0 in G_2 is an exterior point in relation to G_1, so that there is a circle $|w - w_0| < \rho$ belonging to the exterior of G_1. By building the function

$$\frac{1}{w-w_0}=\frac{1}{F_1(z)-w_0}=f(z)$$

we find that it is univalent in G and bounded ($|f(z)| < 1/\rho$).

Hence, in all cases there are functions in G that are univalent and bounded. Suppose that $f(z)$ is such a function, and z_0 is a fixed finite point of G. Then $f'(z_0) \neq 0$ (in view of the univalence of $f(z)$) and, hence,

$$F(z)=\frac{f(z)-f(z_0)}{f'(z_0)}$$

is bounded and univalent in G, vanishes at z_0, and has a first derivative equal to unity at this point.

We denote the set of functions that possess these properties in G by E_{z_0}. Each function $F(z) \in E_{z_0}$ has a finite upper bound for its modulus in G:

$$M(F)=\sup_G |F(z)| > 0.$$

The geometric meaning of $M(F)$ is that it is the radius of the smallest circle centered at the origin of coordinates that contains the image of G in the mapping $w = F(z)$. For different functions $F(z) \in E_{z_0}$ the numbers $M(F)$ are, in general, also different.

We wish to show that there exists an $f(z)$ in E_{z_0} for which $M(F)$ attains its minimal value

$$R_{z_0}=\inf_{F(z)\in E_{z_0}} M(F).$$

The function then conformally maps G into the circle $|w| < R_{z_0}$.

Consider the sequence of functions $F_n(z) \in E_{z_0}$ for which

$$\lim_{n\to\infty} M(F_n) = \inf_{F(z)\in E_{z_0}} M(F) = R_{z_0}$$

(such a sequence exists by the very definition of a lower bound). This sequence is uniformly bounded in G since the converging sequence $\{M(F_n)\}$ is bounded:

$$M(F_n) = \sup_G |F_n(z)| < M \quad (n = 1, 2, \ldots).$$

Montel's theorem states that $\{F_n(z)\}$ is compact in G, so that from this sequence we can select another sequence $\{F_{n_k}(z)\}$ that uniformly converges inside G to an analytic function $f(z)$. For this function we have

$$f(z_0) = \lim_{k\to\infty} F_{n_k}(z_0) = 0 \quad \text{and} \quad f'(z_0) = \lim_{k\to\infty} F'_{n_k}(z_0) = 1,$$

which implies that $f(z) \not\equiv$ const. Applying Hurwitz's theorem (Sec. 8.2), we see that $f(z)$ is univalent in G. Indeed, if for a w_0 the equation $f(z) - w_0 = 0$ has at least two roots in G, the equation $F_{n_k}(z) - w_0 = 0$ must also have more than one root in G, which violates the univalence of $F_{n_k}(z)$.

Next, from

$$f(z) = \lim_{k\to\infty} F_{n_k}(z) \quad \text{and} \quad |F_{n_k}(z)| \leqslant M(F_{n_k}) \to R_{z_0}$$

it follows that, for any positive ε,

$$|f(z)| < |F_{n_k}(z)| + \frac{\varepsilon}{2} < R_{z_0} + \varepsilon$$

for all n_k that are sufficiently large, whence

$$|f(z)| \leqslant R_{z_0} \quad (z \in G)$$

due to the fact that ε is arbitrarily small. But because $f(z)$ is univalent and bounded in G and satisfies the conditions

$$f(z_0) = 0 \quad \text{and} \quad f'(z_0) = 1,$$

we see that $f(z) \in E_{z_0}$ and $\sup_G |f(z_0)| \geqslant R_{z_0}$.

Comparing the two inequalities, we have

$$\sup_G |f(z)| = R_{z_0}.$$

Now we wish to show that the function $w = f(z)$ conformally maps G into the circle K_0: $|w| < R_{z_0}$. Since the properties of $f(z)$ established earlier imply that the image $f(G)$ of G belongs to K_0, it suffices to show that each point in K_0 belongs to $f(G)$. Suppose that this is not so. Then there are boundary points of $f(G)$

inside K_0. We denote one of these points by w_0 $(0 < | w_0 | < R_{z_0})$ and build the sequence

$$w = f(z), \quad w_1 = R_{z_0}^2 \frac{w - w_0}{R_{z_0}^2 - \bar{w}_0 w} = f_1(z), \quad w_2 = \sqrt{R_{z_0} w_1} = f_2(z)$$

(we fix one of the branches of the last function),

$$w_3 = R_{z_0}^2 \frac{w_2 - f_2(z_0)}{R_{z_0}^2 - \overline{f_2(z_0)}\, w_2} = f_3(z), \quad \text{and} \quad w_4 = \frac{w_3}{f_3'(z_0)}.$$

Each of the functions $w_j = f_j(z)$, thought of as a function of w_{j-1} $(w_0 = w)$, is univalent in $f_{j-1}(G)$ $(f_0(G) = f(G))$. For this reason all these functions, considered as functions of z, are univalent in G.

The function $w_1(w)$ maps the circle K_0 into itself, so that point w_0 is mapped into the origin of coordinates. The domain $f(G)$ is mapped into $f_1(G)$ belonging to K_0, and point $w_1 = 0$ is a boundary point of $f_1(G)$. The function $w_2(w_1)$ has two branches in $f_1(G)$. If we fix one of these branches and note that the values on it belong to K_0, with $w_1 = 0$ becoming $w_2 = 0$, we find that $f_2(G)$ also belongs to K_0, and point $w_2 = 0$ lies on the boundary of $f_2(G)$. The function $w_3(w_2)$ maps K_0 into itself, so that point $f_2(z_0)$ is mapped into the origin of coordinates. Hence, the univalent function $f_3(z)$ vanishes at z_0 and maps G into a domain $f_3(G)$ belonging to K_0. Its derivative at point z_0 is

$$f_3'(z_0) = \frac{dw}{dz}\bigg|_{z=z_0} \frac{dw_1}{dw}\bigg|_{w=f(z_0)=0} \frac{dw_2}{dw_1}\bigg|_{w_1=f_1(z_0)=-w_0} \frac{dw_3}{dw_2}\bigg|_{w_2=f_2(z_0)=\sqrt{-R_{z_0}w_0}}$$

$$= 1 \times \frac{R_{z_0}^2 - |w_0|^2}{R_{z_0}^2} \times \frac{\sqrt{R_{z_0}}}{2\sqrt{-w_0}} \times \frac{R_{z_0}^2}{R_{z_0}^2 - R_{z_0}|w_0|} = \frac{R_{z_0} + |w_0|}{2\sqrt{-R_{z_0}w_0}}.$$

From the fact that $| w_0 | < R_{z_0}$ it follows that $| f_3'(z_0) | > 1$. Whence $f_3(z)$ does not belong to E_{z_0}. Dividing it by $f_3'(z_0)$, we find the function $f_4(z)$, which belongs to E_{z_0} (the function $f_4(z)$ is univalent in G and meets the following restrictions: $f_4(z_0) = 0$ and $f_4'(z_0) = 1$). But this function maps G into $f_4(G)$, which belongs to the circle $| w_4 | < \dfrac{R_{z_0}}{| f_3'(z_0) |} < R_{z_0}$; therefore,

$$\sup_G | f_4(z) | \leqslant \frac{R_{z_0}}{| f_3'(z_0) |} < R_{z_0},$$

which contradicts the definition of R_{z_0}, the lower bound of the set of the numbers $\sup\limits_{z \in G} | F(z) |$, $F(z) \in E_{z_0}$. The proof is complete.

We have found that the function performing conformal mapping of a domain into a circle can be subjected to additional constraints:

$$f(z_0) = 0 \quad \text{and} \quad f'(z_0) = 1,$$

where z_0 is an arbitrary finite point of the domain.

The circle K_0 that we have used has its center at the origin of coordinates and is of a definite radius R_{z_0}. The radius is called the *conformal radius of domain G* with respect to point z_0. If instead of $f(z)$ we take $F(z) = (1/R_{z_0}) f(z)$, we have achieved the mapping of G into the unit circle $|w| < 1$. The function that performs the mapping satisfies the following conditions:

$$F(z_0) = 0 \quad \text{and} \quad F'(z_0) > 0.$$

From the geometric viewpoint this means that point $z_0 \in G$ is mapped into the center of the unit circle and that the tangents to curves passing through point z_0 do not undergo rotations in the process of transformation to the curve images, which pass through the center of the circle.

Let us now consider a theorem proved by J. H. Poincaré in 1886.

The uniqueness theorem of the conformal mapping. *There can be only one function $w = f(z)$ that conformally maps a domain G into the unit circle K: $|w| < 1$ and satisfies the conditions $f(z_0) = 0$ and $f'(z_0) > 0$, where z_0 is a fixed point in G.*

Indeed, let $w_1 = f_1(z)$ and $w_2 = f_2(z)$ be two such functions. Obviously, the inverse $z = \varphi_2(w_2)$ to $w_2 = f_2(z)$ conformally maps the circle $|w_2| < 1$ into G so that $\varphi_2(0) = z_0$ and $\varphi_2'(0) > 0$. Whence $w_1 = f_1[\varphi_2(w_2)]$ maps the circle $|w_2| < 1$ into itself in a way such that $f_1[\varphi_2(0)] = 0$ and $\left.\frac{dw_1}{dw_2}\right|_{w_2=0} = \left.\frac{dw_1}{dz}\right|_{z=z_0} \left.\frac{dz}{dw_2}\right|_{w_2=0} = f_1'(z_0)/f_2'(z_0) > 0$.

If we note that Schwarz's lemma (Sec. 10.2) can be applied to $f_1[\varphi_2(w_2)]$ (since $|f_1[\varphi_2(w_2)]| < 1$), we see that $0 < f_1'(z_0)/f_2'(z_0) \leqslant 1$.

In the above reasoning the functions $f_1(z)$ and $f_2(z)$ can be interchanged, which yields $0 < f_2'(z_0)/f_1'(z_0) \leqslant 1$. Comparing these two results, we find that $\left.\frac{dw_1}{dw_2}\right|_{w_2=0} = f_1'(z_0)/f_2'(z_0) = 1$, which in view of Schwarz's lemma is possible only if $w_1 \equiv e^{i\alpha} w_2$, where in the case at hand the factor $e^{i\alpha}$ must be unity $\left(\text{since } \left.\frac{dw_1}{dw_2}\right|_{w_2=0} = 1\right)$. Hence, $w_1 = f_1(z) \equiv w_2 = f_2(z)$, which completes the proof.

10.7 CORRESPONDENCE BETWEEN BOUNDARIES

Here is a corollary of Riemann's mapping theorem: *any two simply connected domains G_1 and G_2 in the extended plane, with boundaries Γ_1 and Γ_2 consisting of more than one point each, can always be conformally mapped into each other.* Indeed, if $w = f_1(z_1)$ maps G_1 into the circle $|w| < 1$ and $w = f_2(z_2)$ maps G_2 into the same circle,

then $z_2 = \varphi_2 [f_1 (z_1)]$, where $z_2 = \varphi_2 (w)$ is the inverse of $w = f_2 (z_2)$, maps G_1 into G_2. Let us assume that the boundaries Γ_1 and Γ_2 of the domains G_1 and G_2 are Jordan curves (in the generalized sense, i.e., perhaps, passing through the point at infinity) and that $z_2 = F (z_1)$ is a univalent function that conformally maps G_1 into G_2. This function is well-defined, obviously, only at the interior points of G_1. It can be proved (but we will take it for granted), however, that to each point $\zeta_1 \in \Gamma_1$ there corresponds a definite limit of $F (z_1)$ as z_1 tends to ζ_1 remaining inside G_1 in the process; this limit ζ_2 lies on Γ_2:

$$\lim_{z_1 \to \zeta_1,\ z_1 \in G_1} F (z_1) = \zeta_2 \quad (\zeta_1 \in \Gamma_1,\ \zeta_2 \in \Gamma_2).$$

Starting with this fact, we can extend the definition of $F (z_1)$ to the closed domain $\overline{G}_1$, assuming that $F (\zeta_1) = \lim\limits_{z_1 \to \zeta_1,\ z_1 \in G_1} F (z_1)$. A function defined in this way is continuous in the extended sense in $\overline{G}_1$ and, in particular, is continuous in the extended sense on Γ_1. Let us assume that this is not the case and $F (z)$ is not continuous at a point $\zeta_1^{(0)} \in \Gamma_1$; for the sake of definiteness we take $\zeta_1^{(0)} \neq \infty$ and $F (\zeta_1^{(0)}) \neq \infty$. Then in a neighborhood $| z - \zeta_1^{(0)} | < \varepsilon$ of point $\zeta_1^{(0)}$ there is a point $\zeta_1 \in \Gamma_1$ such that $| F (\zeta_1^{(0)}) - F (\zeta_1) | \geqslant \alpha > 0$, where α is a constant. But in the same neighborhood there is a point $z_1 \in G_1$ that is so close to ζ_1 that $| F (\zeta_1) - F (z_1) | < \alpha/2$ and, hence, $| F (\zeta_1^{(0)}) - F (z_1) | \geqslant \alpha/2 > 0$. Since the radius of the ε-neighborhood is arbitrarily small, our result contradicts the definition of $F (\zeta_1^{(0)})$ as $\lim\limits_{z_1 \to \zeta_1^{(0)},\ z_1 \in G_1} F (z_1)$.

In the above reasoning the domains G_1 and G_2 can be interchanged, whence for the inverse $z_1 = \Phi (z_2)$ of $z_2 = F (z_1)$ we find that

$$\lim_{z_2 \to \zeta_2,\ z_2 \in G_2} \Phi (z_2) = \zeta_1 \quad (\zeta_2 \in \Gamma_2,\ \zeta_1 \in \Gamma_1)$$

exists.

Assuming that $\Phi (\zeta_2) = \zeta_1$, we find that $\Phi (z)$ is continuous in the extended sense in $\overline{G}_2$ and, in particular, on Γ_2. We can easily see that $\zeta_2 = F (\zeta_1)$ and $\zeta_1 = \Phi (\zeta_2)$ are mutually inverse. These functions establish a one-to-one continuous in the extended sense in both directions correspondence between the Jordan curves Γ_1 and Γ_2.

We have therefore arrived at

The theorem on the correspondence between boundaries (C. Carathéodory, E. Study, and P. Koebe; 1913). *A function $z_2 = F (z_1)$ that conformally maps a domain G_1 bounded by a (closed) Jordan curve Γ_1 into a domain G_2 bounded by a (closed) Jordan curve Γ_2 can be defined (in a unique manner, obviously) on Γ_1 in a way such that it becomes continuous in the extended sense in the closed domain $\overline{G}_1$*

and establishes a one-to-one mapping of Γ_1 into Γ_2, continuous in the extended sense in both directions. In a shorter form: when two domains bounded by (closed) Jordan curves are conformally mapped into each other, their boundaries always correspond in a one-to-one and mutally continuous manner.

We have already noted, we will not prove this theorem because of the complexity of the proof.*

Let us establish the argument principle in its general form, as an important application of the above theorem.

The generalized argument principle. Let $f(z)$ be a function continuous in the extended sense (see Sec. 2.3) in the closed domain $\bar{G}$, where G is the interior of a (closed) Jordan curve γ, and analytic in G everywhere except at possible poles. If on γ the function $f(z)$ does not vanish or becomes infinite, the difference between the number of poles and zeros of $f(z)$ in G is equal to the variation of Arg $f(z)$ in the course of a single complete traversal of γ in the positive sense, divided by 2π, i.e. is equal to the number of loops of the directed curve $\Gamma = f(\gamma)$ around point $w = 0$, taken with the appropriate sign.

Compared with the argument principle given in Sec. 8.2, the generalized argument principle does not require, first, that γ be rectifiable and, second, that $f(z)$ be analytic on γ.

We will start the proof by noting that the number of zeros (N) and the number of poles (P) in G are finite numbers, otherwise on γ there would exist an accumulation point for the set of zeros or poles at which $f(z)$ would vanish or become infinite, respectively.

Suppose that the function $z = \varphi(t)$ conformally maps the unit circle $|t| < 1$ into G. In view of the one-to-one manner of mapping, the function $f(z)$ becomes $f^*(t) = f[\varphi(t)]$, the latter having the same number of zeros and poles in G. Next, in view of the theorem on the correspondence between boundaries, the function $f^*(t)$ is continuous (in the extended sense) in the closed circle $|t| \leqslant 1$, and to one complete traversal of the circle $|t| = 1$ by point t there corresponds one complete traversal of the curve γ by point z. The sense of traversal is discussed below.

Let us assume that the circle $|t| < r_0 < t$ contains all the zeros and poles of $f^*(t)$. Then in view of the argument principle in its former form, the vector $f^*(t)$ makes $N - P$ complete rotations about point $w = 0$ as point t traverses the circle $|t| = r$ $(r_0 < < r < 1)$ once completely in the positive sense. This means that

$$N - P = \frac{1}{2\pi} \operatorname*{Var}_{0 \leqslant \alpha \leqslant 2\pi} \operatorname{Arg} f^*(re^{i\alpha})$$

* See, for example, A. I. Markushevich, *The Theory of Analytic Functions* [in Russian], vol. 2, Nauka, Moscow, 1968, Chap. 5, § 3.

(α increases). We introduce the notation $\rho = \min\limits_{0 \leqslant \alpha \leqslant 2\pi} |f^*(e^{i\alpha})|$; since $f^*(e^{i\alpha}) = f(z)$, $z \in \gamma$, we find that ρ is positive. In view of the uniform continuity of $f^*(t)$ in the closed annulus $r_0 \leqslant |t| \leqslant 1$ (here we do not mean continuity in the extended sense because $f^*(t) \neq \infty$ in the annulus) there is an r_1, with $r_0 \leqslant r_1 < 1$, such that

$$|f^*(re^{i\alpha}) - f^*(e^{i\alpha})| < \rho, \quad 0 \leqslant \alpha \leqslant 2\pi, \quad r_1 \leqslant r \leqslant 1.$$

We write $f^*(re^{i\alpha})$ in the form

$$f^*(re^{i\alpha}) = f^*(e^{i\alpha})[1 + \lambda_r(\alpha)],$$

where

$$|\lambda_r(\alpha)| = \frac{|f^*(re^{i\alpha}) - f^*(e^{i\alpha})|}{|f^*(e^{i\alpha})|} < 1 \quad \text{for} \quad r_1 \leqslant r \leqslant 1.$$

Whence

$$\operatorname*{Var}_{0 \leqslant \alpha \leqslant 2\pi} \operatorname{Arg}[1 + \lambda_r(\alpha)] = 0,$$

and, hence, for $r \geqslant r_1 \geqslant r_0$ we have

$$N - P = \frac{1}{2\pi} \operatorname{Var} \operatorname{Arg} f^*(re^{i\alpha}) = \frac{1}{2\pi} \operatorname*{Var}_{0 \leqslant \alpha \leqslant 2\pi} \operatorname{Arg} f^*(e^{i\alpha})$$

(α increases).

Returning to the z plane by means of the conformal mapping $z = \varphi(t)$, we find that

$$N - P = \pm \frac{1}{2\pi} \operatorname*{Var}_{\gamma} \operatorname{Arg} f(z),$$

where γ is traversed by point z in the positive sense. Here the sign on the right-hand side does not depend on the function $f(z)$; we must take either $+$ or $-$ depending on whether the sense of traversal is preserved or changes to the opposite in the event of the conformal mapping of G into the unit circle (or, in general, into a bounded domain with a Jordan curve as boundary). To decide what sign to choose we put $f(z) = z - z_0$, where $z_0 \in G$. Here $N = 1$ and $P = 0$, and

$$\frac{1}{2\pi} \operatorname*{Var}_{\gamma} \operatorname{Arg}(z - z_0) = 1$$

(γ is traversed in the positive sense). Then from the proved relationship we conclude that the sign on the right-hand side must be $+$. Hence

$$N - P = \frac{1}{2\pi} \operatorname*{Var}_{\gamma} \operatorname{Arg} f(z),$$

which completes the proof.

At the same time we have arrived at an important addition to the theorem on the correspondence between boundaries:

The correspondence between boundaries established in the conformal mapping $z_2 = f(z_1)$ *of a domain* G_1 *into a domain* G_2 *is always such that point* $z_2 = f(z_1)$ *traverses the boundary of* G_1 *in the positive sense if* z_1 *traverses the boundary of* G_2 *in the same sense.*

There is a theorem that follows from the generalized argument principle and that is the converse of the theorem on the correspondence between boundaries. However, it is valid only under certain restrictions. We take its simpler case where the Jordan curves Γ_1 and Γ_2 do not pass through the point at infinity and G_1 and G_2 are the interiors of these curves. This theorem was proved by Ch. E. Picard in 1905.

The converse theorem on the correspondence between boundaries. *Let* $z_2 = F(z_1)$ *be a function continuous in a closed domain* $\overline{G}_1$ *bounded by a Jordan curve* Γ_1, *and analytic in* G_1. *If it establishes a one-to-one mapping, continuous in both directions, of* Γ_1 *into a Jordan curve* Γ_2, *the function is univalent in* G_1 *and maps this domain into the interior* G_2 *of* Γ_2.

The theorem can easily be extended to include the case where G_1 is the exterior of Γ_1, or where Γ_1 is a generalized Jordan curve and G_1 is one of the two domains bounded by this curve (for instance, Γ_1 a straight line and G_1 a half-plane). To reduce each of these cases to the one above, it suffices to fix a point $z_1^{(0)}$ in the exterior of G_1 and proceed with the auxiliary mapping $z' = 1/(z_1 - z_1^{(0)})$. As a result Γ_1 is mapped into a Jordan curve Γ' in the z' plane (the curve does not pass through point $z' = \infty$), G_1 into a domain G' (the interior of Γ'), and $F(z_1)$ into the function $F(z_1^{(0)} + 1/z')$, which is continuous in the closed domain $\overline{G'}$, analytic in G', and establishes a one-to-one mapping of Γ' into Γ_2, continuous in both directions. We note, however, that Γ_1 and Γ_2 do not play an equal role in the hypothesis of the theorem, and the theorem becomes invalid if Γ_2 passes through the point at infinity. The simplest example in this respect is the function $z_2 = z_1^3$ in the upper half-plane. Obviously, it maps the boundary of the half-plane, the real axis, in a one-to-one and mutually continuous (in the extended sense) manner into the real axis, too. But the function is not univalent in the upper half-plane and maps the latter not into the upper half-plane but into a Riemann surface that can be obtained by taking two copies G_2' and G_2'' of the upper half-plane and one copy G_2''' of the lower half-plane, and gluing G_2' and G_2'' along the negative half of the real axis and G_2'' and G_2''' along the positive half of the real axis (Fig. 72).

Now we turn to the proof of the theorem. It will be based on the generalized argument principle. Suppose that $z_2^{(0)}$ is a point in G_2.

We wish to show that this point belongs to the set of values of $F(z_1)$ in G_1, and that the equation $F(z_1) - z_2^{(0)} = 0$ has one and only one root in G_1. By the argument principle the number of roots, N, in this equation is $\frac{1}{2\pi} \underset{z_1 \in \Gamma_1}{\text{Var Arg}} [F(z_1) - z_2^{(0)}]$ on the assumption that z_1 traverses Γ_1 once completely in the positive sense. But, by the hypothesis of the theorem $z_2 = F(z_1)$ must in the process traverse Γ_2 in a certain sense. Therefore, the vector $F(z_1) - z_2^{(0)} = z_2 - z_2^{(0)}$ with the initial point at $z_2^{(0)}$ and the terminal point on Γ_2 and which lies completely in the interior of Γ_2 must turn through an angle of $\pm 2\pi$; whence

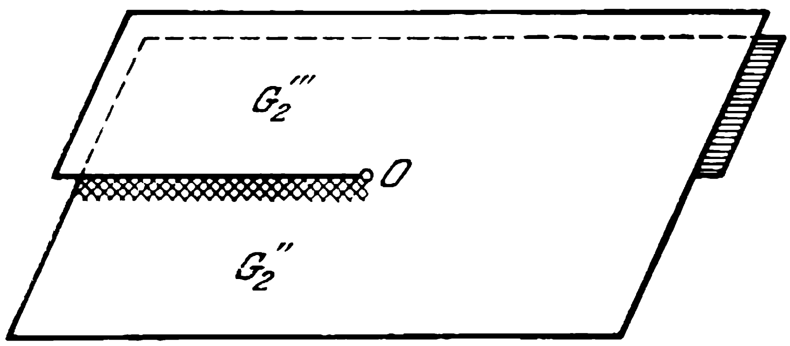


Fig. 72

$$N = \frac{1}{2\pi} \underset{z_1 \in \Gamma_1}{\text{Var Arg}} [F(z_1) - z_2^{(0)}] = \frac{1}{2\pi} \underset{z_2 \in \Gamma_2}{\text{Var Arg}} (z_2 - z_2^{(0)}) = \pm 1.$$

The minus sign is obviously impossible ($N \geqslant 0$). We therefore conclude that point $z_2 = F(z_1)$ must move along Γ_2 in the positive sense (in relation to the interior of Γ_2) and the equation $F(z_1) - z_2^{(0)} = 0$ has one root in G_1. In a similar manner we can show that for a point z_2' lying in the exterior of Γ_2 the equation $F(z_1) - z_2' = 0$ has not a single root in $G_1 \left(\frac{1}{2\pi} \underset{z_1 \in \Gamma_1}{\text{Var Arg}} [F(z_1) - z_2'] = 0 \right)$.

We have proved that the set of values of $F(z_1)$ in G_1 contains all points lying in the interior of Γ_2 and no points in the exterior. Since this set is a domain (each point is an interior point), it coincides with the domain G_2 bounded by Γ_2. Finally, each point $z_2 \in G_2$ is a value of $F(z_1)$ attained at only one point in G_1. Whence $z_2 = F(z_1)$ is univalent in G_1 and maps this domain conformally into G_2.

For applications, a very important case is when G_1 is the upper half-plane and the function $F(z_1)$ has a finite number of singular points on the real axis while remaining continuous in $\bar{G}_1$ (see below Sec. 10.10).

On the basis of the fact of correspondence between boundaries, we can subject the function $z_2 = f(z_1)$, which conformally maps

G_1 into G_2, to additional restrictions of a type different from that mentioned in Sec. 10.6. We can require, for example, that point $z_1^{(0)} \in G_1$ becomes $z_2^{(0)} \in G_2$ in the mapping and, in addition, the boundary point $\zeta_1^{(0)}$ becomes the boundary point $\zeta_2^{(0)}$; nothing can be said, however, about $\operatorname{Arg} f'(z_1^{(0)})$. Similarly, we can require that any three points ζ_1', ζ_1'', ζ_1''' on Γ_1 become the three points ζ_2', ζ_2'', ζ_2''' on Γ_2, the only restriction being that the sense of traversal is preserved. To verify the validity of the two statements it suffices to consider the case where, say, the domain G_2 is the interior of the unit circle. If a conformal mapping of G_1 into G_2 does not satisfy the requirements, for instance if the images of ζ_1', ζ_1'', ζ_1''' on the unit circle do not coincide with the preassigned, we have only to conformally map the unit circle into itself to get the desired result.

10.8

MAPPING THE UPPER HALF-PLANE BY MEANS OF AN ELLIPTIC INTEGRAL

As an application of the converse theorem on the correspondence between boundaries we will study the mapping of the upper half-plane by the function

$$w = f(z) = \int_0^z \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}},$$

where $0 < k^2 < 1$. This function is known as *Legendre's normal elliptic integral of the first kind*; k is known as the *modulus* (*module*) of the elliptic integral. The function

$$\sqrt{(1-t^2)(1-k^2t^2)}$$

is double-valued and has branch points of multiplicity 2 at $t = \pm 1$ and $t = \pm 1/k$. Since all lie on the real axis, in the upper half-plane we can isolate two branches of this function that differ only in sign at each point. We take the branch that assumes positive values on the interval (0, 1) of the real axis. Then in the upper half-plane the integral represents a single-valued analytic function that is continuous in the closed half-plane and analytic everywhere except at points ± 1 and $\pm 1/k$. (In the neighborhood $|z| > 1/|k|$ of the point at infinity and integrand can be expanded in the following series:

$$\begin{aligned}(1-t^2)^{-1/2}(1-k^2t^2)^{-1/2} &= k^{-1}t^{-2}(1-t^{-2})^{-1/2}(1-k^{-2}t^{-2})^{-1/2} \\ &= \pm k^{-1}t^{-2}\left(1+\frac{1}{2}t^{-2}+\dots\right)\left(1+\frac{1}{2k^2}t^{-2}+\dots\right) \\ &= \pm\left[k^{-1}t^{-2}+\frac{1}{2}(k^{-1}+k^{-3})t^{-4}+\dots\right].\end{aligned}$$

For $f(z)$ we then have

$$f(z) = C_0 \mp \left[k^{-1}z^{-1} + \frac{1}{2\times 3}(k^{-1}+k^{-3})z^{-3} + \dots \right],$$

which shows that both branches of $f(z)$ are analytic functions at the point at infinity.)

Taking the upper half-plane as the domain G_1 and the real axis as the boundary Γ_1 (we noted above that the extension of the theorem to the case of unbounded domains and curves is legitimate), we consider the mapping of Γ_1 by the function $w = f(z)$. When $z = x$ traverses the segment $0 < x < 1$, the function

$$w = f(x) = \int_0^x \frac{dt}{\underset{+}{\sqrt{(1-t^2)(1-k^2t^2)}}}$$

remains real and increases from zero to

$$K = \int_0^1 \frac{dt}{\underset{+}{\sqrt{(1-t^2)(1-k^2t^2)}}}.$$

On the segment $1 < x < 1/k$ the integrand is

$$\frac{1}{\pm i \underset{+}{\sqrt{(t^2-1)(1-k^2t^2)}}},$$

where the sign cannot be chosen arbitrarily; it must correspond to the above choice of the branch of the square root to ensure the continuity of this branch in the upper half-plane. If we write $(1-t^2)\times$ $\times(1-k^2t^2)$ in the form

$$k^2(t-1)[t-(-1)]\left(t-\frac{1}{k}\right)\left[t-\left(-\frac{1}{k}\right)\right] = \varphi(t),$$

we note that when we go over from the points in the segment $(0, 1)$ to those in $(1, 1/k)$ on the semicircle with the center at point 1 in the upper half-plane, the variation of $\operatorname{Arg}\varphi(t)$, which is a result of variations of the arguments of separate factors, is equal to $-\pi$ (namely, $\operatorname{Arg}(t-1)$ decreases by π, while the arguments of the other factors do not change). For this reason $\operatorname{Arg}\sqrt{\varphi(t)}$ changes by $-\pi/2$ and, hence, takes on the value $-\pi/2 + 2k\pi$. We see that of the two values $\pm i \underset{+}{\sqrt{(t^2-1)(1-k^2t^2)}}$ we must take $-i\underset{+}{\sqrt{(t^2-1)(1-k^2t^2)}}$. Hence, for $1 < x < 1/k$ we have

$$w = f(x) = \int_0^x \frac{dt}{\underset{+}{\sqrt{(1-t^2)(1-k^2t^2)}}} = \int_0^1 \frac{dt}{\underset{+}{\sqrt{(1-t^2)(1-k^2t^2)}}}$$

$$+ i\int_1^x \frac{dt}{\underset{+}{\sqrt{(t^2-1)(1-k^2t^2)}}} = K + i\int_1^x \frac{dt}{\underset{+}{\sqrt{(t^2-1)(1-k^2t^2)}}},$$

which implies that when $z = x$ traverses the segment $1 < x < 1/k$, point w traverses a straight segment parallel to the imaginary axis from point K to point $K + iK'$, with

$$K' = \int_1^{1/k} \frac{dt}{\underset{+}{\sqrt{(t^2-1)(1-k^2t^2)}}} .$$

This integral can be written in a form similar to that of K if we introduce the substitution

$$t = \frac{1}{\sqrt{1-k'^2t'^2}}, \text{ where } k'^2 = 1-k^2.$$

This yields

$$K' = \int_0^1 \frac{dt}{\underset{+}{\sqrt{(1-t^2)(1-k'^2t^2)}}} .$$

If we go over from the segment $1 < x < 1/k$ to the segment $1/k < x < +\infty$, the argument of the integrand

$$(t^2-1)(1-k^2t^2) = -k^2(t-1)(t+1)\left(t-\frac{1}{k}\right)\left(t+\frac{1}{k}\right)$$

decreases by π (together with Arg $(t-1/k)$). For this reason for $\sqrt{(t^2-1)(1-k^2t^2)}$ we obtain the value with argument $-\pi/2$:

$$-i \underset{+}{\sqrt{(t^2-1)(k^2t^2-1)}}.$$

Hence, for $1/k < x < +\infty$

$$w = f(x) = \int_0^1 \frac{dt}{\underset{+}{\sqrt{(1-t^2)(1-k^2t^2)}}} + i\int_1^{1/k} \frac{dt}{\underset{+}{\sqrt{(t^2-1)(1-k^2t^2)}}} - \int_{1/k}^{x} \frac{dt}{\underset{+}{\sqrt{(t^2-1)(k^2t^2-1)}}}$$

$$= K + iK' - \int_{1/k}^{x} \frac{dt}{\underset{+}{\sqrt{(t^2-1)(k^2t^2-1)}}} .$$

If x increases from $1/k$ to $+\infty$, the value of the integral $\int_{1/k}^{x} \frac{dt}{\underset{+}{\sqrt{(t^2-1)(k^2t^2-1)}}}$ increases from zero to

$$\int_{1/k}^{\infty} \frac{dt}{\underset{+}{\sqrt{(t^2-1)(k^2t^2-1)}}} = \int_0^1 \frac{d\tau}{\underset{+}{\sqrt{(1-\tau^2)(1-k^2\tau^2)}}} = K,$$

where we have introduced the substitution $t = 1/k\tau$. Whence point $w = f(x)$ traverses a straight segment parallel to the imaginary axis from $K + iK'$ to iK'.

In a similar manner we can show that when x traverses the segments from 0 to -1, from -1 to $-1/k$, and from $-1/k$ to $-\infty$, point $w = f(x)$ successively traverses straight segments from 0 to $-K$, from $-K$ to $-K + iK'$, and, finally, from $-K + iK'$ to iK'.

Thus, *the function* $w = f(z)$ *maps in a one-to-one manner the real axis* Γ_1 *into the contour* Γ_2 *of a rectangle with vertices at* $-K$, K, $K + iK'$, *and* $-K + iK'$, *where*

$$K = \int_0^1 \frac{dt}{\underset{+}{\sqrt{(1-t^2)(1-k^2t^2)}}}, \quad K' = \int_0^1 \frac{dt}{\underset{+}{\sqrt{(1-t^2)(1-k'^2t^2)}}},$$

and $k'^2 = 1 - k^2$.

Then from the above-proved theorem it follows that this function is univalent in the upper half-plane and maps it conformally into the rectangle mentioned above.

The base of the rectangle is $2K$ and its height is K', whence

$$\frac{K'}{2K} = \frac{\displaystyle\int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k'^2t^2)}}}{\displaystyle 2\int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}} = \lambda(k)$$

If the parameter $k\,(0 < k < 1)$ increases from 0 to 1, the denominator increases from $2\displaystyle\int_0^1 \frac{dt}{\sqrt{1-t^2}} = \pi$ to ∞; the process k' $(k' = \underset{+}{\sqrt{1-k^2}})$ drops from 1 to 0 and, hence, the numerator decreases from ∞ to $\pi/2$. Whence, $K'/2K = \lambda(k)$ continuously decreases from ∞ to 0 as k increases from 0 to 1. Therefore, for any rectangle of base $2a$ and height b there is one and only one value of k $(0 < k < 1)$ for which

$$\frac{b}{2a} = \frac{\displaystyle\int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k'^2t^2)}}}{\displaystyle 2\int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}} = \frac{K'}{2K}.$$

Consequently, the function

$$f(z) = \int_0^z \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}$$

built for this value of k conformally maps the upper half-plane into a rectangle similar to the given one.

If we want to obtain a mapping into the given rectangle, we have only to introduce the factor $\mu = 2a/2K = b/K'$. We find that

$$w = \mu \int_0^z \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}\,.$$

As a result of the mapping we arrive at a rectangle with vertices $-\mu K = -a$, $\mu K = a$, $\mu K + i\mu K' = a + bi$, $-\mu K + i\mu K' = -a + bi$.

Thus, *by means of the elliptic integral*

$$\mu \int_0^z \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}$$

we can map the upper half-plane into a rectangle with preassigned sides by fitting the parameters k and μ.

10.9

JACOBI'S ELLIPTIC FUNCTION SN w. INVERSION OF ULTRAELLIPTIC INTEGRALS

Here we will dwell on the properties of the function that is the inverse of the elliptic integral of the first kind:

$$w = \int_0^z \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}} = f(z) \quad (0 < k < 1).$$

As we saw in Sec. 10.8, the function $w = f(z)$ conformally maps the upper half-plane into the rectangle Δ_0 with vertices $\pm K$ and $\pm K + iK'$ (Fig. 73), and this function is continuous in the closed upper half-plane and maps the real axis in a one-to-one and mutually continuous manner into the contour of the rectangle so that the points $0, \pm 1, \pm 1/k, \infty$ become $0, \pm K, \pm K + iK', iK'$, respectively. This implies that the inverse function, which is called *Jacobi's elliptic function* and denoted by $z = \operatorname{sn}(w; k)$ (or $z = \operatorname{sn} w$ in short form), is single-valued and univalent in Δ_0, continuous (in the extended sense) in $\bar{\Delta}_0$, and maps in a one-to-one and mutually

continuous manner the contour of the rectangle into the real axis, so that the base AB becomes the segment $[-1, +1]$, the sides BC and DA become the segments $[1, 1/k]$ and $[-1/k, -1]$, and finally, the sides CE and ED of the upper base become the segments $[1/k, \infty]$ and $[\infty, -1/k]$.

Obviously, $\operatorname{sn} w$ can be continued analytically across AB, BC, DA, CE, and DE according to the Riemann-Schwarz symmetry principle (Sec. 9.5). For instance, across AB the function is continued on the rectangle Δ_1: AD_1C_1B. By the symmetry principle

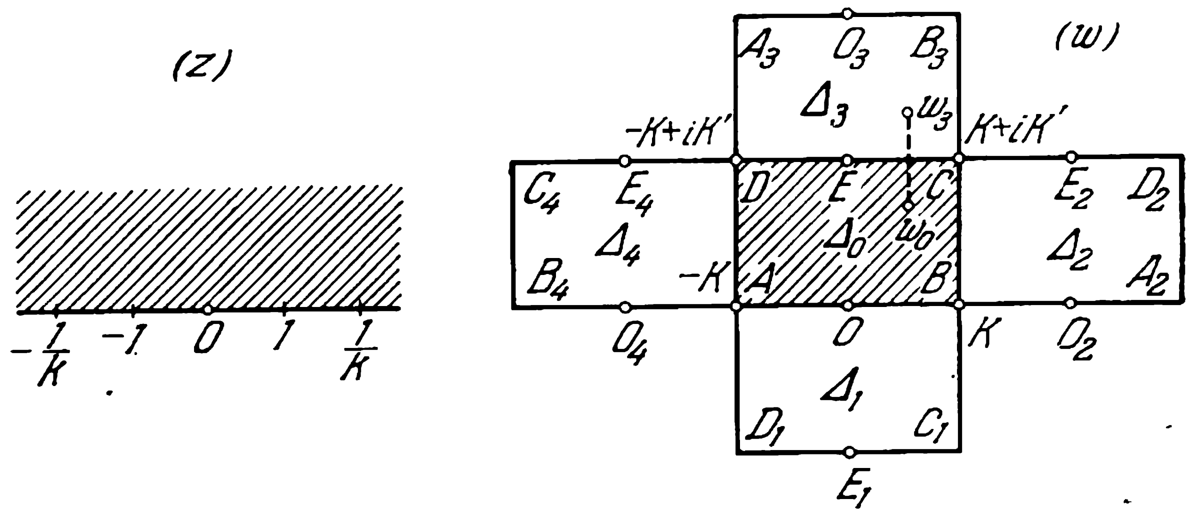


Fig. 73

the function $z = \operatorname{sn} w$ in this rectangle attains values that belong to the lower half-plane. Due again to the symmetry of continuation, it is univalent in Δ_1, maps the sides BC_1 and AD_1 into the segments $[1, 1/k]$ and $[-1/k, -1]$, and C_1E_1 and E_1D_1 into the segments $[1/k, \infty]$ and $[\infty, -1/k]$. Note that point 0 is a zero of $\operatorname{sn} w$; and this zero is simple because in the rectangle DD_1C_1C the function is univalent.

We make similar statements for the analytic continuations across the sides BC or DA of the rectangle Δ_0. Since in the middle E of the upper base of this rectangle the function $\operatorname{sn} w$ becomes infinite, continuation may be achieved only across CE and ED. We can easily see that in both cases the result is continuation into the same rectangle Δ_3 and we arrive at the same function. Specifically, if point $w_3 \in \Delta_3$ is symmetric to $w_0 \in \Delta_0$ with respect to the straight line with CE and ED, in both cases we have $\operatorname{sn} w_3 = \overline{\operatorname{sn} w_0}$. Hence continuation of $\operatorname{sn} w$ from Δ_0 across CE and ED brings us to a function $\operatorname{sn} w$ that is single-valued and analytic in the rectangle ABB_3A_3 everywhere except at point E with the affix iK', at which it becomes infinite. Consequently, iK' is a pole of $\operatorname{sn} w$. But the function is univalent inside the rectangle ABB_3A_3, whence point iK' is a simple pole (Sec. 10.5). After we have continued our function into the rectangles Δ_1, Δ_2, Δ_3, Δ_4 we can repeat the process and use the

symmetry principle to continue it analytically to still other rectangles. The infinite process will result in a function sn w that is continued into the entire finite w plane as a single-valued analytic function without any singularities except an infinitude of simple poles. Hence, the elliptic function $z = \operatorname{sn} w$ is meromorphic. As a result of the continuation process the entire plane will be covered by a network of equal rectangles placed in a regular pattern, the rectangles being alternately continued (analytically) by the function $z = \operatorname{sn} w$ to the upper or lower half-plane. Part of this network

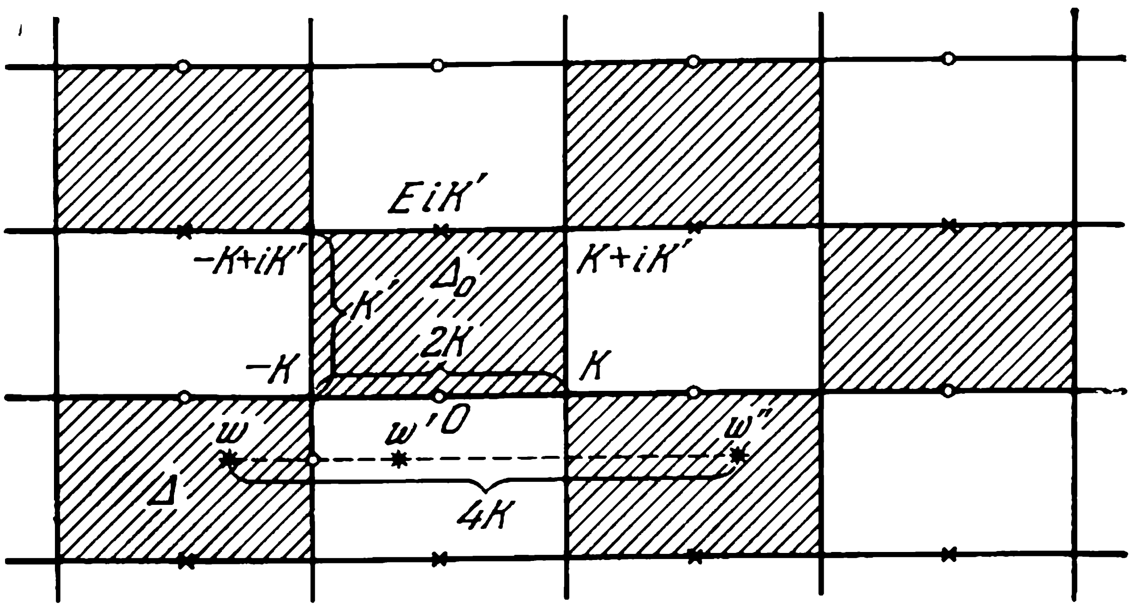


Fig. 74

is depicted in Fig. 74, where the rectangles continued to the upper half-plane are hatched. Small circles depict the zeros of sn w and crosses depict the poles (the position of both, as shown earlier, follows directly by the symmetry principle, from the data on these points established for sn w in Δ_0). Obviously, all the zeros of sn w are given by the expression $2mK + 2niK'$ and all the poles by $2mK + (2n + 1)\, iK'$ (m and n are arbitrary integers); the order of both poles and zeros is unity, i.e. both are simple.

Finally, let us see whether sn w is periodic, namely, has periods of the type $4mK + 2niK'$, where m and n are arbitrary integers. This can easily be verified if we find that $4K$ and $2iK'$ are periods, since integral multiples of these two numbers will then definitely be periods (and so will sums of these integral multiples). Consider Fig. 74; suppose that w is a point on one of the rectangles. We apply the symmetry principle twice, each time continuing sn w across the right side of the appropriate rectangle. We find that $\operatorname{sn} w' = \overline{\operatorname{sn} w}$ and $\operatorname{sn} w'' = \overline{\operatorname{sn} w'}$, whence $\operatorname{sn} w'' = \operatorname{sn} w$. Obviously, the segment with end points at w and w'' is parallel to the real axis and is twice as long as the base of the rectangle; whence $w'' = w + 4K$. Thus,

$\mathrm{sn}\,(w+4K) = \mathrm{sn}\,w$ (for any w). Similarly, using the symmetry principle twice so that each time the continuation takes place across the upper side of the appropriate rectangle, we find that $\mathrm{sn}\,(w+2iK') = \mathrm{sn}\,w$.

We have therefore established that the function $z = \mathrm{sn}\,w$, which is meromorphic and possesses two independent periods $4K$ and $2iK'$, agrees completely with the definition of an elliptic function given in Sec. 8.6. Like Weierstrass's $\wp(w)$ function, this is a second-order elliptic function. But in the period parallelogram it has two simple poles, while $\wp(w)$ has one pole of order 2.

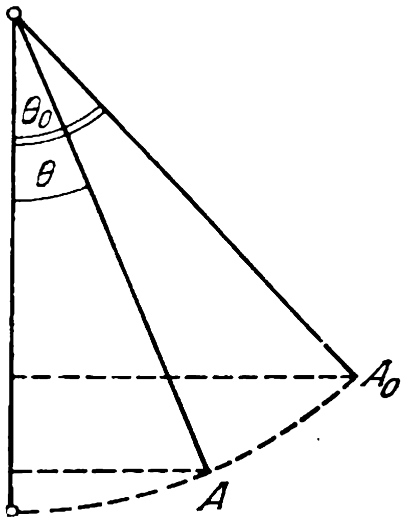


Fig. 75

Note that for real values of the independent variable, $w = u$, the function $\mathrm{sn}\,w$ is encountered in the problem of the oscillations of a simple pendulum. The problem consists in studying the motion of a small weight suspended from a point by a light thread; the oscillations take place in a vertical plane (Fig. 75).

Let l be the length of the thread, m the mass of the weight (a ball), g the acceleration of free fall, θ the angle of deflection of the thread from the vertical at time t, and θ_0 the maximal value of θ. Comparing the position of the weight at point A_0, where its velocity is zero, with that at point A, where its velocity is $v = l\,(d\theta/dt)$, we find from the balance of energies that

$$\frac{mv^2}{2} = mg\,(l\cos\theta - l\cos\theta_0);$$

whence, substituting for v its value $l\,(d\theta/dt)$ and integrating, we have

$$t = \sqrt{\frac{l}{2g}}\int_0^\theta \frac{d\theta}{\sqrt{\cos\theta-\cos\theta_0}} = \frac{1}{2}\sqrt{\frac{l}{g}}\int_0^\theta \frac{d\theta}{\sqrt{\sin^2\frac{\theta_0}{2}-\sin^2\frac{\theta}{2}}}$$

(the starting moment, $t = 0$, is when $\theta = 0$). Introducing a new variable τ by the relationship $\sin(\theta/2) = \tau\sin(\theta_0/2)$, we obtain

$$t = \sqrt{\frac{l}{g}}\int_0^1 \frac{d\tau}{\sqrt{(1-\tau^2)\left(1-\sin^2\frac{\theta_0}{2}\tau^2\right)}},$$

whence

$$\tau = \mathrm{sn}\left(\sqrt{\frac{g}{l}}\,t;\ \sin\frac{\theta_0}{2}\right), \text{ or } \sin\frac{\theta}{2} = \sin\frac{\theta_0}{2}\,\mathrm{sn}\left(\sqrt{\frac{g}{l}}\,t;\ \sin\frac{\theta_0}{2}\right).$$

This formula reduces the simple pendulum problem to a study of the behavior of the elliptic function sn $(u;\ k)$ at $k=\sin\ (\theta_0/2)$. Since the real period of an elliptic function is $4K=4\int\limits_0^1 \frac{d\tau}{\sqrt{(1-\tau^2)\left(1-\sin^2\frac{\theta_0}{2}\tau^2\right)}}$, the oscillation period of the simple pendulum will obviously be $4K/\sqrt{g/l}$, i.e.

$$4\sqrt{\frac{l}{g}}\int\limits_0^1 \frac{d\tau}{\sqrt{(1-\tau^2)\left(1-\sin^2\frac{\theta_0}{2}\tau^2\right)}}.$$

Historically, the problem of inverting elliptic integrals naturally led to the problem of inverting *ultraelliptic integrals*, i.e. integrals of the type $\int\limits_{z_0}^{z} R\,(z, \sqrt{P\,(z)})\,dz$, where $P\,(z)$ is a polynomial of a degree higher than four (with simple zeros). The simplest of these integrals, considered by Jacobi in 1834, is

$$w=\int\limits_0^z \frac{(\alpha+\beta t)\,dt}{\sqrt{t\,(1-t)\,(1-\varkappa t)\,(1-\lambda t)\,(1-\mu t)}},\ \text{where}\ 0<\mu<\lambda<\varkappa<1. \tag{10.8}$$

Studying this integral, Jacobi arrived at the strange conclusion that the inverse function $z=\lambda\,(w)$ is absurd and, hence, it is useless to study the properties of such functions, in contrast to elliptic functions, in the development of the theory of which he and Abel independently played a considerable role. At present we possess the means for establishing the reason for Jacobi's discouraging conclusion.

For the sake of simplicity we take $\alpha=1$ and $\beta=0$. Then, choosing in the interval (0, 1) of the real axis the one of the two branches of the square root that assumes positive values, we can repeat the reasoning of the previous section, at least the main aspects of this reasoning. We introduce the following notations:

$$t\,(1-t)\,(1-\varkappa t)\,(1-\lambda t)\,(1-\mu t)=P\,(t),$$

$$\int\limits_0^1 dt/\underset{+}{\sqrt{P\,(t)}}=A,\quad \int\limits_1^{1/\varkappa} dt/\underset{+}{\sqrt{-P\,(t)}}=B,\quad \int\limits_{1/\varkappa}^{1/\lambda} dt/\underset{+}{\sqrt{P\,(t)}}=C,$$

$$\int\limits_{1/\lambda}^{1/\mu} dt/\underset{+}{\sqrt{-P\,(t)}}=D,\quad \int\limits_{1/\mu}^{+\infty} dt/\underset{+}{\sqrt{P\,(t)}}=E,\quad \int\limits_{-\infty}^{0} dt/\underset{+}{\sqrt{-P\,(t)}}=F.$$

The positive real numbers A, B, C, D, E and F are (in this order) the lengths of the sides of a hexagon with right angles (one of the angles is $3\pi/2$ and the hexagon is not convex) to which the ultraelliptic integral, or rather the branch in the upper half-plane selected according to the above condition, conformally maps the upper half-plane (Fig. 76). We note in passing that there is no need to prove first that, say, $C > A$ and $D < B$. The polygon of the required type is obtained of its own accord if we only note the sense of traversal of the contour by point w as point z traverses the real axis in the direction of increasing z's and the fact that the contour is closed (w tends to the same limit as z tends to $\pm\infty$ and remains on the real axis). As a result we arrive at the following relationships for the six integrals introduced above: $A + E = C$ and $D + F = B$.

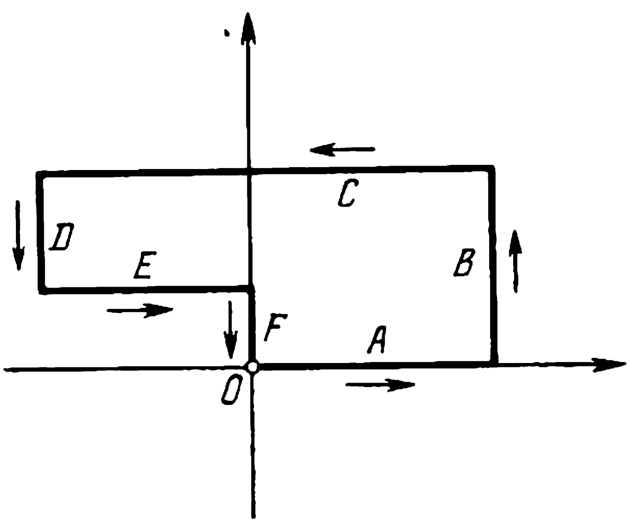


Fig. 76

The function $z = \varphi(w)$, which is the inverse of the given integral, conformally maps the built hexagon to the upper half-plane, $\operatorname{Im} w > 0$. This function can be continued analytically outside the hexagon if the latter is reflected with respect to each of its six sides and the same is done over and over with the resulting hexagons. As a result the entire z plane is covered by an infinitude of layers of hexagonal parquet (this happens in the general case where the numbers $\varkappa$, λ, and μ are not connected via special relationships). In each piece of the parquet there is defined a branch of $z = \lambda(w)$ that maps the appropriate hexagon into the half-plane $\operatorname{Im} w > 0$ or $\operatorname{Im} w < 0$, depending on whether an even or odd number of reflections is required to pass from the initial hexagon to the given one.

More than that, the Riemann-Schwarz symmetry principle makes it possible to prove that the function $z = \varphi(w)$ has four independent periods (the proof is similar to that for the function $z = \operatorname{sn} w$). We can take them to be two real numbers, $2A$ and $2E$, and two pure imaginary, $2iD$ and $2iF$. Any other period Ω of this function is a linear combination of the four:

$$\Omega = 2Am_1 + 2Em_2 + 2Dim_3 + 2Fim_4,$$

where m_1, m_2, m_3, and m_4 are arbitrary integers.

But Jacobi established that a function with four periods must have *infinitely small* nonzero periods. This means that for any positive ε there is a nonzero Ω such that $|\Omega| < \varepsilon$. This implies that the

function $\varphi(w)$ admits each of its values $\varphi(w_0)$ not only at points w_0 but at arbitrarily close points of the type $w_0 + \Omega$, since $\varphi(w_0 + \Omega) = \varphi(w_0)$. This very conclusion led Jacobi to believe that $z = \varphi(w)$ is absurd. And this is quite natural, since he had no way of observing such phenomena in the then known analytic functions, both single-valued (recall the uniqueness theorem) and multiply valued, such as Ln w or Arc tan w. The fact is that Jacobi encountered an extremely complex multiply valued analytic function. The problem is resolved if we take the multilayer parquet as a Riemann surface. Then the infinitude of values of the type $w_0 + \Omega$ are distributed one by one on each piece of parquet belonging to different layers. The complex numbers $w_0 + \Omega$ are indeed very close to w_0, but the image points on the Riemann surface are by no means reduced to one point.

Jacobi's mistake, nevertheless, was productive for mathematics. In the same work he set forth a completely new problem of inverting ultraelliptic integrals—he substituted for one integral the sums of the different values of such integrals. This started the theory of *Abel's functions*, which is an important topic in the theory of analytic functions of several complex variables.

10.10

THE SCHWARZ-CHRISTOFFEL TRANSFORMATION FORMULA

The following transformation formula is a generalization of the idea of the elliptic integral considered in Sec. 10.8.

$$w = f(z) = C \int_0^z (t-a_1)^{\alpha_1-1} (t-a_2)^{\alpha_2-1} \dots (t-a_n)^{\alpha_n-1}\, dt. \tag{10.9}$$

Here C is a positive real number (say $C = 1$), the numbers $a_1, a_2, \ldots, a_n$ are different and real (for the sake of definiteness we assume that $a_1 < a_2 < \ldots < a_n$), and the exponents $\alpha_1 - 1, \ldots, \alpha_n - 1$ are real but some may be equal. The elliptic integral of Sec. 10.8 can be obtained if we put $n = 4$, $a_1 = -1/k$, $a_2 = -1$, $a_3 = 1$, $a_4 = 1/k$, and $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 1/2$.

Returning to the general case, we subject the exponents $\alpha_j - 1$ to additional restrictions that ensure the convergence of the *Schwarz-Christoffel integral* (10.9) at each point a_k and at $z = \infty$. Here are these restrictions:

$$\alpha_j - 1 > -1 \ (j = 1, 2, \ldots, n) \text{ and } \alpha_1 + \ldots + \alpha_n - n < -1. \tag{10.10}$$

(the last restriction follows from the fact that in a neighborhood of point $z = \infty$ the integrand can be written as

$$z^{\alpha_1+\ldots+\alpha_n-n}\left(1-\frac{a_1}{z}\right)^{\alpha_1-1}\ldots\left(1-\frac{a_n}{z}\right)^{\alpha_n-1}\Big).$$

We take a definite branch $\lambda(z) = (z-a_1)^{\alpha_1-1}\ \ldots\ (z-a_n)^{\alpha_n-1}$ in the upper half-plane by subjecting it to the following condition: on the part of the real axis where $z = x < a_1$ and, hence, all the differences $z - a_j$ are negative, we take for each difference the value of the argument equal to π, while the argument of $\lambda(z)$ we take equal to $(\alpha_1-1)\pi + \ldots + (\alpha_n-1)\pi$. If we agree to this, in the intervals (a_{k-1}, a_k) we must ascribe to the argument of $\lambda(z)$ values that agree with the chosen one. It suffices to bear in mind

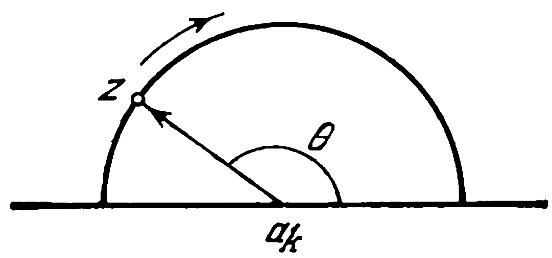


Fig. 77

that when point z goes over from the interval (a_{k-1}, a_k) to (a_k, a_{k+1}) $(k = 1, 2, \ldots, n,\ a_0 = \infty,$ and $a_{n+1} = \infty)$ only one difference $z - a_j$ changes sign (from $-$ to $+$), namely, $z - a_k$. If we make z traverse a semicircle with its center at a_k belonging to the upper half-plane (Fig. 77), we find that the argument of $z - a_k$ must decrease by π and, hence, vanishes. Correspondingly, the argument of $(z-a_k)^{\alpha_k-1}$ must change by $-(\alpha_k-1)\pi$ and so must the argument of $\lambda(z)$. Since in the interval (∞, a_1) the latter has the value $(\alpha_1-1)\pi+\ldots+(\alpha_n-1)\pi$ in the interval (a_1, a_2) it increases its value by $-(\alpha_1-1)\pi$ and then keeps a constant value $(\alpha_2-1)\pi + \ldots + (\alpha_n-1)\pi$; in general, in the interval (a_{k-1}, a_k) $(k = 2, 3, \ldots, n)$ it has a value of $(\alpha_k-1)\pi + \ldots$ $\ldots + (\alpha_n-1)\pi$ and, finally, in the interval (a_n, ∞) a value of 0. Assuming that

$$C\int_0^{a_{k-1}}(t-a_1)^{\alpha_1-1}\ldots(t-a_n)^{\alpha_n-1}\,dt = w_k,$$

let us consider the behavior of the integral (10.9) in the interval (a_{k-1}, a_k). We have

$$w = f(z) = w_{k-1} + C\int_{a_{k-1}}^{z}\lambda(t)\,dt$$

$$= w_{k-1} + Ce^{(\alpha_k - 1)\pi i + \ldots + (\alpha_n - 1)\pi i} \int\limits_{a_{k-1}}^{z} |\lambda(t)|\, dt. \qquad (10.11)$$

The second term on the right-hand side preserves its argument $(\alpha_k - 1)\pi + \ldots + (\alpha_n - 1)\pi$, and its modulus continuously increases from zero to $l_k = C \int\limits_{a_{k-1}}^{a_k} |\lambda(t)|\, dt$, as $z = x$ increases from a_{k-1} to a_k. This implies that when z increases and traverses the interval (a_{k-1}, a_k), the point $w = f(z)$ traverses in one direction a straight segment Δ_{k-1} with the beginning at point w_{k-1}, sloped at an angle $(\alpha_k - 1)\pi + \ldots + (\alpha_n - 1)\pi$ to the positive direction of the real axis, and with a length equal to l_k; the end point of this segment is at point $w_k = C \int\limits_0^{a_k} (t - a_1)^{\alpha_1 - 1} \ldots$ $\ldots (t - a_n)^{\alpha_n - 1}\, dt$. This is also true with respect to the interval (∞, a_1) (here the beginning of segment Δ_0 lies at $w_0 = C \int\limits_0^{\infty} (t - a_1)^{\alpha_1 - 1} \ldots (t - a_n)^{\alpha_n - 1}\, dt$) and the interval (a_n, ∞) (here the end point of Δ_n lies at

$$w_{n+1} = C \int\limits_0^{\infty} (t - a_1)^{\alpha_1 - 1} \ldots (t - a_n)^{\alpha_n - 1}\, dt).$$

The above reasoning implies that *the function* (10.9) *maps the real axis into the closed broken line* Λ *with sections* $\Delta_0, \ldots, \Delta_n$ *and vertices at* $w_0, w_1, \ldots, w_n, w_{n+1} = w_0$. But we cannot say that this mapping is one-to-one since the broken line may have self-intersections, i.e. the segments Δ_j may have common points otherwise than at the vertices. Suppose that there are no self-intersections (as was the case, for instance, with the elliptic integral). Then Λ is a Jordan curve bounding a polygon with vertices at $w_0, \ldots, w_n$; hence, according to Sec. 10.7, the Schwarz-Christoffel integral (10.9) conformally maps the upper half-plane into the given polygon. From Fig. 78 it follows that the interior angle of this polygon at a vertex w_k $(0 < k < n + 1)$ is equal to $\alpha_k\pi$. Although each angle depicted in this figure must be considered as defined to within an integral multiple of 2π, we can show that $\alpha_k\pi$ is exactly an interior angle of the polygon; in other words, $0 < \alpha_k < 2$, while (10.10) requires that $\alpha_k > 0$ only. To verify this, we use the fact that the mapping

$w = f(z)$ is univalent in a neighborhood of point a_k. We note that

$$\lambda(z) = [(z-a_1)^{\alpha_1-1} \dots (z-a_{k-1})^{\alpha_{k-1}-1}(z-a_{k+1})^{\alpha_{k+1}-1} \times \dots \times (z-a_n)^{\alpha_n-1}](z-a_k)^{\alpha_k-1},$$

where the expression in the brackets is a function analytic and nonzero in a neighborhood of point a_k. This expression can, therefore, be expanded in a series in powers of $z - a_k$ with a nonzero

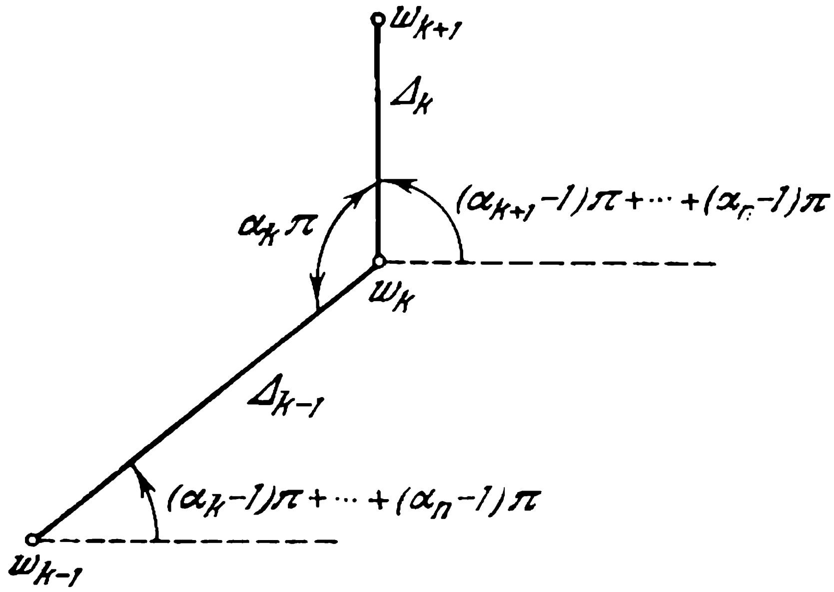


Fig. 78

absolute term: $A_0^{(k)} + A_1^{(k)}(z - a_k) + \dots$, and this series is uniformly convergent in a neighborhood of a_k. But then in this neighborhood

$$w = w_k + C\int_{a_k}^{z} \lambda(t)\,dt$$

$$= w_k + C\int_{a_k}^{z} [A_0^{(k)} + A_1^{(k)}(t-a_k) + \dots](t-a_k)^{\alpha_k-1}\,dt$$

$$= w_k + C\frac{A_0^{(k)}}{\alpha_k}(z-a_k)^{\alpha_k} + C\frac{A_1^{(k)}}{\alpha_k+1}(z-a_k)^{\alpha_k+1} + \dots \quad (A_0^{(k)} \neq 0),$$

whence

$$w - w_k = C\frac{A_0^{(k)}}{\alpha_k}(z-a_k)^{\alpha_k}\left[1 + \frac{\alpha_k}{\alpha_k+1}\frac{A_1^{(k)}}{A_0^{(k)}}(z-a_k) + \dots\right].$$

If z tends to a_k along a radius of a circle centered at a_k that makes an angle θ $(0 < \theta < \pi)$ with the positive direction of the real axis (see Fig. 77), the above formula implies that w will tend to point w_k along a curve whose tangent at point w_k has an inclination angle equal to $\lim\limits_{w \to w_k} \operatorname{Arg}(w - w_k) = \operatorname{Arg} A_0^{(k)} + \alpha_k\theta$ (we have al-

lowed here for C and α_k being positive). If θ changes continuously from 0 to π, the radii emerging from point a_k also change continuously and fill up a semicircle with its center at a_k. The image curves, i.e. the above-mentioned curves, will also behave in a similar manner—they will fill the entire interior angle of the polygon at the vertex w_k (close to the vertex). But the inclination angle of the tangents to these curves at point w_k will change by $\alpha_k\pi$ (from $\operatorname{Arg} A_0^{(k)}$ to $\operatorname{Arg} A_0^{(k)} + \alpha_k\pi$); whence $\alpha_k\pi$ is the magnitude of the interior angle at w_k and, therefore, $0 < \alpha_k < 2$ $(k = 1, 2, \ldots, n)$.

For the vertex $w_0 = w_{n+1}$ the magnitude of the interior angle proves to be

$$[(n-1)-(\alpha_1 + \ldots + \alpha_n)]\,\pi$$

(Fig. 79). From the inequality (10.10) we can only conclude that this is a positive number. But if we recall that our polygon has only $n+1$ angles and therefore the angle sum must be $[(n+1)-2]\,\pi = (n-1)\,\pi$, knowing the magnitudes of the other n angles, namely $\alpha_1\pi, \ldots, \alpha_n\pi$ for the $(n+1)$st angle, we have exactly

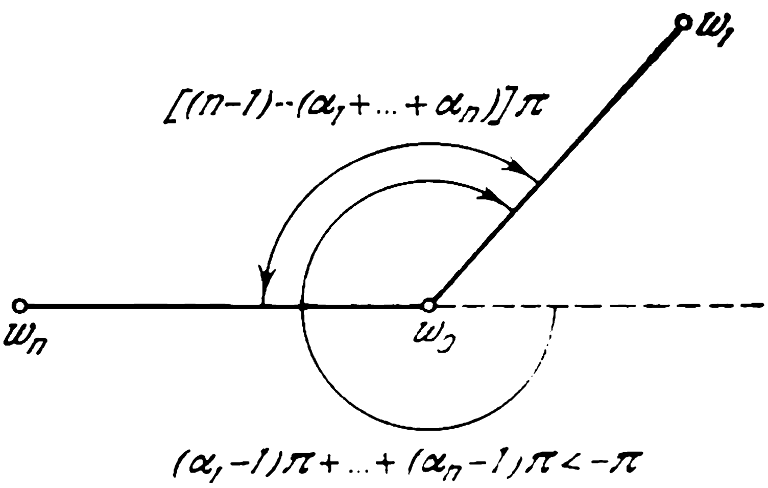


Fig. 79

$$[(n-1)-(\alpha_1 + \ldots + \alpha_n)]\,\pi.$$

Whence we have

$$0 < [(n-1)-(\alpha_1 + \ldots + \alpha_n)]\,\pi < 2\pi,$$

which implies that

$$\alpha_1 + \alpha_2 + \ldots + \alpha_n > n-3.$$

Thus, the numbers α_k $(k = 1, 2, \ldots, n)$ must satisfy the following inequalities

$$0 < \alpha_k < 2 \quad (k = 1, \ldots, n), \qquad n-3 < \alpha_1 + \ldots + \alpha_n < n-1. \qquad (10.10')$$

These conditions are necessary for the Schwarz-Christoffel integral to conformally map the upper half-plane into a bounded polygon with vertices w_k and interior angles $\alpha_k\pi$ $(k = 1, 2, \ldots, n)$ and $[(n-1)-(\alpha_1 + \ldots + \alpha_n)]\,\pi$. But these are not sufficient conditions.

In the particular case where $\alpha_1 + \ldots + \alpha_n = n-2$ we find that the size of the angle at w_0 is π. Geometrically this means that w_0 is not a vertex but simply one of the interior points of the side with vertices w_n and w_1. In other words, in this case the integral

(10.9) maps the upper half-plane into an n-gon instead of a $(n+1)$ gon. This is the case for the elliptic integral, where $n = 4$ and $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 4 \times 1/2 = 4 - 2$.

Let us use the Schwarz-Christoffel integral to map the upper half-plane into a triangle with angles $\alpha_1\pi$, $\alpha_2\pi$, and $\alpha_3\pi$ ($\alpha_k > 0$, $\alpha_1 + \alpha_2 + \alpha_3 = 1$) and side of length l opposite to the angle $\alpha_1\pi$. There are two ways in which we can proceed. First, we can use (10.9) at $n = 3$, taking three arbitrary real numbers a_1, a_2, and a_3 as points on the real axis that must be mapped into the vertices of the triangle (see the last paragraph in Sec. 10.7). For example, we take $a_1 = -1$, $a_2 = 0$, and $a_3 = 1$. We have

$$w = C \int_0^z (t+1)^{\alpha_1 - 1} t^{\alpha_2 - 1} (t-1)^{\alpha_3 - 1}\, dt.$$

We still have to find the positive factor C. To this end we note that according to the aforesaid l must be equal to

$$C \int_0^1 |(t+1)^{\alpha_1 - 1} t^{\alpha_2 - 1} (t-1)^{\alpha_3 - 1}|\, dt$$

$$= C \int_0^1 (1+t)^{\alpha_1 - 1} t^{\alpha_2 - 1} (1-t)^{\alpha_3 - 1}\, dt,$$

whence

$$C = l \Big/ \int_0^1 (1+t)^{\alpha_1 - 1} t^{\alpha_2 - 1} (1-t)^{\alpha_3 - 1}\, dt.$$

Thus

$$w = l \frac{\int_0^z (t+1)^{\alpha_1 - 1} t^{\alpha_2 - 1} (t-1)^{\alpha_3 - 1}\, dt}{\int_0^1 (1+t)^{\alpha_1 - 1} t^{\alpha_2 - 1} (1-t)^{\alpha_3 - 1}\, dt}. \tag{10.12}$$

This function solves the problem because it maps the real axis into a triangle with the three given angles and one given side (the triangle has no self-intersections). In particular, when $\alpha_1 = \alpha_2 = \alpha_3 = 1/3$, we have an equilateral triangle and the function assumes the form

$$w = l \frac{\int_0^z \frac{dt}{\sqrt[3]{t^2 (t^2 - 1)^2}}}{\int_0^1 \frac{dt}{\sqrt[3]{t^2 (1 - t^2)^2}}};$$

for $\alpha_2 = 1/2$ and $\alpha_1 = \alpha_3 = 1/4$ we have a mapping into a right isosceles triangle with a leg of length l:

$$w = l \frac{\int\limits_0^z \frac{dt}{\sqrt{t}\sqrt[4]{(t^2-1)^3}}}{\int\limits_0^1 \frac{dt}{\sqrt{t}\sqrt[4]{(1-t^2)^3}}};$$

etc.

Second, we can solve the same problem of mapping the upper half-plane into a triangle by taking $n = 2$ in (10.9) and selecting two points on the real axis a_1 and a_2 that must be mapped into two vertices of the triangle with angles $\alpha_1\pi$ and $\alpha_2\pi$.

Since $\alpha_1 + \alpha_2$ is not $n - 2$ $(= 0)$, the point at infinity must be mapped into the third vertex, where the angle is $[(2 - 1) - (\alpha_1 + \alpha_2)]\,\pi = \pi - \alpha_1\pi - \alpha_2\pi = \alpha_3\pi$. For example, if we take $a_1 = 0$ and $a_2 = 1$, the mapping function becomes

$$w = C \int\limits_0^z t^{\alpha_1 - 1} (t-1)^{\alpha_2 - 1}\, dt,$$

where C can be determined from the equation

$$C \int\limits_1^\infty |t^{\alpha_1 - 1} (t-1)^{\alpha_2 - 1}|\, dt = l,$$

which leads to

$$C = l \Big/ \int\limits_1^\infty t^{\alpha_1 - 1} (t-1)^{\alpha_2 - 1}\, dt.$$

Consequently, our problem is also solved by

$$w = l \frac{\int\limits_0^z t^{\alpha_1 - 1} (t-1)^{\alpha_2 - 1}\, dt}{\int\limits_1^\infty t^{\alpha_1 - 1} (t-1)^{\alpha_2 - 1}\, dt}, \tag{10.13}$$

which obviously differs from (10.12).

For example, for mapping the upper half-plane into an equilateral triangle with side l we have the function

$$w = l \frac{\int\limits_0^z \frac{dt}{\sqrt[3]{t^2(t-1)^2}}}{\int\limits_1^\infty \frac{dt}{\sqrt[3]{t^2(1-t)^2}}}.$$

In the general case, when we must map the real axis into an n-gon ($n \geqslant 4$) with angles $\alpha_1\pi, \ldots, \alpha_n\pi$ and lengths $l_1, l_2, \ldots, l_n$ of the sides (angles $\alpha_{k-1}\pi$ and $\alpha_k\pi$ belong to l_k ($k = 2, \ldots, n$), and angles $\alpha_n\pi$ and $\alpha_1\pi$ to l_1), we can choose for (10.9) arbitrary points a_1, a_2, and a_3 that must be mapped into the vertices w_1, w_2, and w_3 with angles $\alpha_1\pi$, $\alpha_2\pi$, and $\alpha_3\pi$ (see the last paragraph in Sec. 10.7). Now we have to define the remaining $n-3$ points on the real axis, $a_4, \ldots, a_n$, and the positive constant C—altogether $n-2$ unknowns. At first glance it would seem that the number of equations for determining these unknowns is more than enough: there are n equations expressing the lengths of sides as integrals. But we can easily see that if the angles of the n-gon are known, then $n-2$ given values $l_1, l_2, \ldots, l_{n-2}$ uniquely define the other two sides, l_{n-1} and l_n (Fig. 80). Therefore, there remain only $n-2$ independent equations

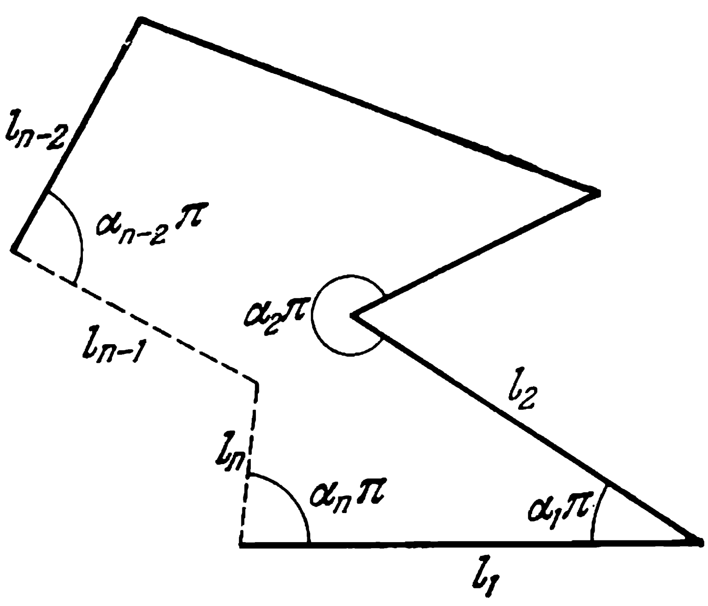


Fig. 80

$$C \int\limits_{a_{k-1}}^{a_k} |(t-a_1)^{\alpha_1-1} \ldots (t-a_n)^{\alpha_n-1}| \, dt = l_k \quad (k=1, 2, \ldots, n-2) \tag{10.14}$$

for determining the $n-2$ unknowns, $a_4, \ldots, a_n, C$.

Using Riemann's mapping theorem and the theorem on the correspondence between boundaries as the basis, we can prove that one can always fit the constants in (10.9) in a way such that (10.9) will map a half-plane into an arbitrary preassigned n-gon. Whence

Eqs. (10.14) will always have solutions. For more details on the question of mapping a half-plane into an arbitrary n-gon we refer the reader to fuller courses on function theory*.

10.11

SCHWARZ'S MODULAR FUNCTION AND PICARD'S FIRST THEOREM

The *modular function*, which is first mentioned in the early unpublished works of Gauss, got its name from its role in the theory of elliptic functions and integrals. In the simplest case it expresses the square of the *modulus* of an elliptic integral k (see Sec. 10.8) in terms of the ratio of periods $\tau = 2iK'/4K$. Here we will consider the purely geometric interpretation of the modular function, which

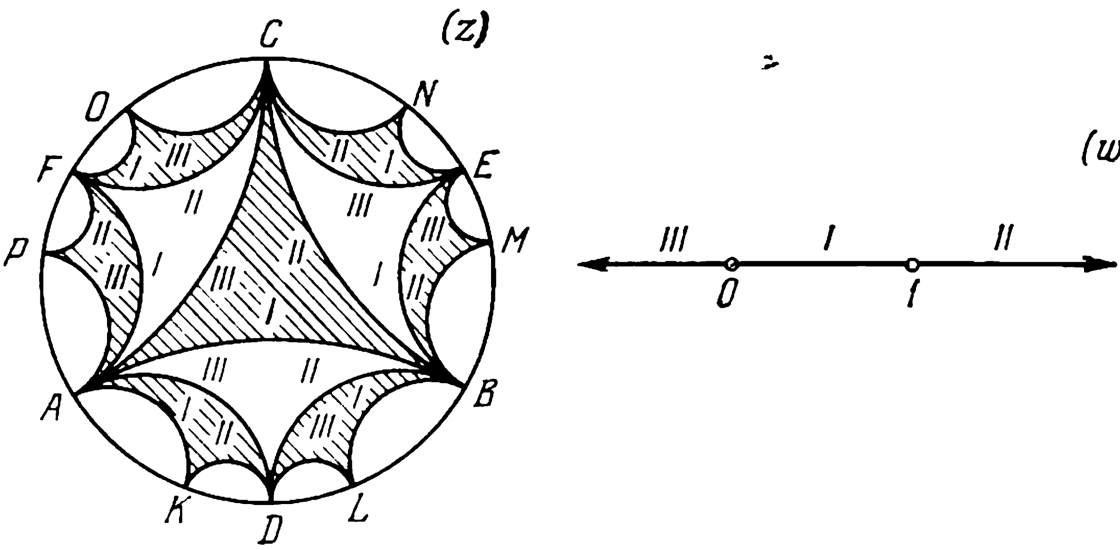


Fig 81

is closely related with the above interpretation. This interpretation was first suggested by H. A. Schwarz in 1873 and the function bears his name.

On the unit circle in the z plane we take three points A, B, and C and connect them pairwise by arcs of circles orthogonal to the unit circle (Fig. 81). These arcs enclose a domain G_0, a triangle with zero angles. Let us map G_0 conformally into the upper half-plane w so that points A, B, and C are mapped into points 0, 1, and ∞, respectively. Then the arcs $\widehat{AB}$, $\widehat{BC}$, and $\widehat{CA}$ become segments $[0, 1]$, $[1, +\infty]$, and $[-\infty, 0]$ on the real axis; we denote them in this order by I, II, and III.

Suppose that $w = \chi(z)$ is a function that executes the mapping. We apply to this function and to G_0 the Riemann-Schwarz symmetry principle (see Sec. 9.5). We find that $\chi(z)$ is analytically continued

* See, for example, A. I. Markushevich, *The Theory of Analytic Functions* [in Russian], vol. 2, Nauka, Moscow, 1968, Chap. 8, § 7.

across each arc, $\overset{\frown}{AB}$, $\overset{\frown}{BC}$, and $\overset{\frown}{CA}$, into the triangles with zero angles, ABD, BCE, and ACF. The values that this function admits in each triangle must belong to the lower half-plane (due to the symmetry principle) and completely fill the latter. Now we apply the symmetry principle to the same function and to each triangle ABD, BCE, and ACF. We find that $\chi(z)$ is analytically continued across their sides AD, DB, BE, EC, CF, and AF into the triangles with zero angles, ADK, DBL, BEM, ECN, CFO, and FAP. In each of these the function admits values that fill the upper half-plane. Repeating this process over and over, we find that $\chi(z)$ can be analytically continued into the entire unit circle.

Indeed, the domain into which the continuation has been carried out after a finite number of steps constitutes a polygon whose sides are arcs of circles orthogonal to the unit circle. First it is the triangle ABC, then the hexagon $ADBECF$, then a dodecagon, and so on (see Fig. 81). We note in passing that the very process of analytic continuation can be carried out by applying the symmetry principle each time not to triangles but to the polygons obtained after each step. Then after triangle ABC we obtain hexagon $ADBECF$, but already after this step the hexagon will undergo the symmetry transformation with respect to each of its six sides and become as a result a $6 + 4 \times 6 = 30$-gon (not given in the figure). If we apply the symmetry principle to this polygon and each of its sides, we arrive at a continuation of $\chi(z)$ into a $30 + 28 \times 30 = 870$-gon. The process can go on indefinitely.

To prove that at each step every point of the circle proves to be inside the appropriate polygon, it suffices to show that the lengths of the arcs of the unit circle on which the sides of the polygon rest (of the two arcs with the same end points we take the shorter one) tend to zero as the number of the polygon sides increases without limit. This in turn follows from the fact that each point on the unit circle is a limit point for the set of vertices of our polygons. Suppose that the contrary is true; then there must exist a sequence of nested arcs $\{\sigma_n\}$ of the unit circle, with the arcs connecting neighboring vertices of polygons, and this sequence tends to an arc σ not lying inside any vertex of the polygon.

We connect the end points of arc σ by an arc τ orthogonal to the unit circle. Implementing the symmetry transformation with respect to τ, we find on σ points A', B', and C' symmetric to A, B, and C (Fig. 82). Reflecting our polygons with respect to their sides that rest on σ_n, we find that the images of the vertices A, B, and C are arbitrarily close to A', B', and C' for sufficiently large n's; hence, starting from a certain value of n at least one vertex belongs to arc σ (and does not coincide with the end points of σ). This contradiction shows that $\chi(z)$ is indeed analytically continued to the entire unit circle. The uniqueness of the continuation follows from

the fact that in each step the reflected polygon lies in the exterior of the circle with respect to which the reflection is performed; hence, this polygon and the reflected one have no common points.

The above construction implies that the function $\chi(z)$ admits in the unit circle any complex number except 0, 1, and ∞ an infinite number of times. The values from the upper half-plane are admitted by this function in the hatched triangles in Fig. 81 and from the lower half-plane in the unhatched ones. The real values

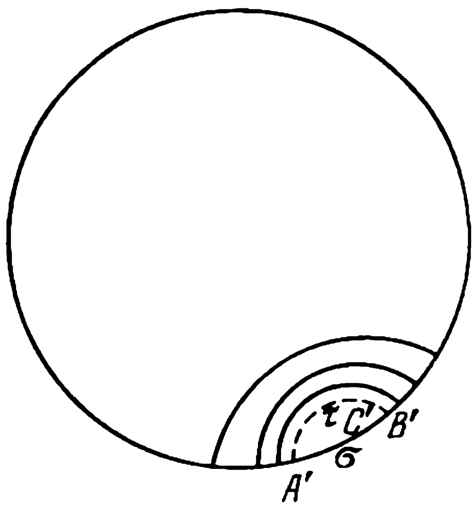


Fig. 82

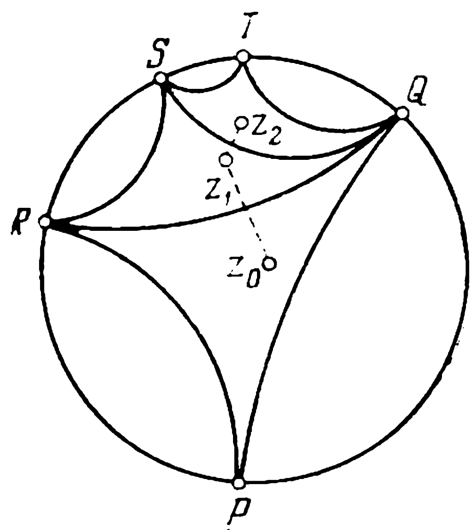


Fig. 83

from the intervals (0, 1), (1, ∞), and ($-\infty$, 0) are admitted on the sides of the triangles denoted by *I*, *II*, and *III* (Fig. 81).

In view of the symmetry principle (see the end of Sec. 9.5), the value $\chi(z_1)$ at a point z_1 obtained by reflecting z_0 with respect to the side QR (Fig. 83) of the triangle PQR (this is one of the images of the triangle ABC obtained as a result of successive reflections with respect to the arcs of circles orthogonal to the unit circle) is conjugate to $\chi(z_0)$, i.e. $\overline{\chi(z_0)} = \chi(z_1)$. A second reflection, for instance with respect to the side QS, leads to a point z_2, and $\chi(z_2) = \overline{\chi(z_1)} = \chi(z_0)$. Thus a double reflection brings the function $\chi(z)$ back to the initial value. But each reflection that is an inversion of the unit circle with respect to an orthogonal circle is expressed by a function that is conjugate to a linear-fractional function (see Sec. 3.8). Whence a double reflection, i.e. the product of two reflections, is represented by a linear-fractional function $z_2 = \frac{az_0+b}{cz_0+d}$ that maps the unit circle into itself. Therefore, $\chi\left(\frac{az_0+b}{cz_0+d}\right) = \chi(z_0)$. Different linear-fractional functions correspond to different pairs of successive reflections. This implies that there must be an infinite sequence

$$\left\{L_n[z] = \frac{a_nz+b_n}{c_nz+d_n}\right\}$$

of linear-fractional functions that map the unit circle into itself (it constitutes a mapping group) such that $\chi\left(\frac{a_n z+b_n}{c_n z+d_n}\right)=\chi(z)$ for every positive integer n and for every point z, $|z|<1$.

From this viewpoint, *Schwarz's modular function* $\chi(z)$ is a particular case of *automorphic functions*. The latter are single-valued analytic functions invariant under linear-fractional mappings (not necessarily mapping a circle into itself). The simplest case is when the mapping is reduced to translations. The result is *periodic* functions, specifically *elliptic*.

The function $w=\chi(z)$ maps the unit circle into a Riemann surface that consists of an infinite number of copies of the upper and lower half-planes. Each upper (lower) half-plane is glued to three lower (upper) half-planes along the segments *I*, *II*, and *III* of the real axis. On this surface the function that is the inverse of the given one is single-valued, analytic, and bounded in its modulus (all its values belong to the unit circle). The same function considered as a function of the complex variable w (we denote it by $\omega(w)$) is multiply valued and analytically continuable along any continuous curve not passing through points 0, 1, and ∞. Among the infinite number of values that $\omega(w)$ admits at a point w there is one and only one that belongs either to the closed triangle ABC or to the interior of the triangle ADB; this value of $\omega(w)$ is called *principal*.

The surface we have just specified is known as the *Riemann surface for the modular function*.

We use the function $\omega(w)$, the inverse of the modular function, to prove

Picard's first theorem. *An entire function* $f(z)\not\equiv$ const *admits any finite complex value except, perhaps, one value.*

Proof. Suppose that $f(z)$ has two finite values that it does not admit, a and b. Then

$$\varphi(z)=\frac{f(z)-a}{b-a}$$

is an entire function that neither vanishes nor becomes equal to unity. Then all values of this function belong to a domain in which a multiply valued function $\omega(w)$ is defined. We fix one branch of the latter in a neighborhood of point $w_0=\varphi(0)$. Then in this neighborhood we have an analytic function element $\omega[\varphi(z)]$, which is analytically continuable along any continuous curve in the finite plane and, hence, by the monodromy theorem (Sec. 9.10), defines a function that is single-valued and analytic in the finite plane, i.e. an entire function. Since the modulus of this function is limited (the values of the function belong to the unit circle), Liouville's theorem states that $\omega[\varphi(z)]\equiv$ const. But this is pos-

sible only if $\varphi(z) \equiv \text{const}$, which implies that $f(z) = \text{const}$. The contradiction proves Picard's first theorem.

We could have also proved that for an entire function $f(z)$ of a finite order ρ, when ρ is not an integer, there can be no exceptional values, i.e. the equation $f(z) = A$ has roots for any complex A. Thus, the exceptional value provided for by Picard's first theorem exists only for functions of integral or infinite order. Here are several simple examples.

(a) $f(z) = e^z$; here $\rho = 1$ and the exceptional value is 0;

(b) $f(z) = e^{e^z}$; here $\rho = \infty$ and the exceptional value is 0, too (note that the value 1 is admitted only at points where $e^z = 2k\pi i$, i.e. $z = \text{Ln}\,(2k\pi i)$, $k = \pm 1, \pm 2, \ldots$);

(c) $f(z) = \sin z$; here $\rho = 1$ and there is no exceptional value (verify this);

(d) $f(z) = \sin \sqrt{z}/\sqrt{z}$; here $\rho = 1/2$ and therefore there is no exceptional value in view of the above theorem (not proved by us).

Picard's first theorem leads to a similar theorem for meromorphic functions.

Theorem. *A meromorphic function* $F(z) \not\equiv \text{const}$ *admits any complex value except, perhaps, two values.*

Proof. Suppose $F(z)$ does not admit three values a, b, and c. If one value, say c, is ∞, then $F(z)$ is an entire function that does not admit two finite values a and b, which by Picard's first theorem is impossible (here $F(z) \not\equiv \text{const}$). Hence, the numbers a, b, and c can be considered finite. We build the function

$$\Phi(z) = \frac{1}{F(z) - c}.$$

This is a meromorphic function that does not become infinite and, hence, is an entire function; besides, $\Phi(z) \not\equiv \text{const}$ and does not admit two finite values $\left(\frac{1}{a-c} \text{ and } \frac{1}{b-c}\right)$. We have once more arrived at a contradiction with Picard's first theorem, which proves the validity of the proposition.

Simple examples of meromorphic functions with two exceptional values are e^z (the exceptional values are 0 and ∞) and $\tan z$ (the exceptional values are $\pm i$). Elliptic functions form a class of meromorphic functions without exceptional values. Indeed, if $f(z)$ is an elliptic function of order p ($p \geqslant 2$), then for any A the equation $f(z) = A$ has exactly p roots in each period parallelogram (see Sec. 8.6).

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